MATHÉMATIQUES

Noyaux de la chaleur, Marches aléatoires, Analyse sur les variétés et les graphes

S. Semmes

Introduction au texte

Le trimestre du Centre Emile Borel intitulé Noyaux de la chaleur, Marches aléatoires, Analyse sur les variétés et les graphes *a eu lieu à l'Institut Henri Poincaré du 16 avril au 13 juillet 2002. Les organisateurs scientifiques en étaient Pascal Auscher (Amiens), Gérard Besson (Grenoble I), Thierry Coulhon (Cergy-Pontoise), et Alexander Grigor'yan (Imperial College). Outre le CEB et l'IHP, le Réseau TMR Harmonic Analysis and Related Problems, l'Université de Cergy-Pontoise, l'Université de Picardie-Jules Verne, et l'École Doctorale Économie et Mathématiques Paris-Ouest ont apporté leur soutien.*

Stephen Semmes (Rice University) a bien voulu rédiger un compte-rendu de ce trimestre, accompagné d'un court survey qui donnera une idée plus précise de certains des sujets qui ont été abordés.

Nous donnons ci-dessous la liste des principaux événements scientifiques du trimestre.

Colloques :

Harmonic Analysis and PDE's (22-26 avril 2002)
en collaboration avec le TMR HARP

Heat kernels and analysis on manifolds (28-31 mai 2002)

Analysis on graphs and metric spaces (26-28 juin 2002)

Cours :

J-P. Anker : *The heat kernel on symmetric (and related) spaces*

P. Auscher et P. Tchamitchian : *Elliptic operators in divergence form*

M. Barlow : *Diffusions on fractals*

R. Brooks : *Spectral geometry of Riemann surfaces*

T. Coulhon et A. Grigory'an : *Isoperimetry and heat kernels on Riemannian manifolds*

B. Driver : *Heat kernels and infinite dimensional analysis*

P. Hajlasz : *Sobolev spaces on metric spaces*

W. Hebisch : *Heat kernels on Lie groups*

S. Hofmann et A. McIntosh : *Functional calculus and the square root problem of Kato*

I. Holopainen : *Quasi-conformal mappings and p-Laplace operator*
V. Maz'ya : *Sobolev spaces and elliptic equations on bad domains*
L. Saloff-Coste : *Heat kernels on infinite dimensional locally compact spaces*
S. Semmes : *Basic topics in analysis and geometry on metric spaces*
K.-T. Sturm : *Non linear heat flows and harmonic maps into metric spaces*
T. Sunada : *Spectral geometry of crystal lattices*
W. Woess : *Generating function techniques for random walks on graphs*
Serge Lang a donné une série de conférences sur Séries d'Eisenstein tordues par le noyau de la chaleur*.*

Nous avons la tristesse de rappeler que notre collègue et ami Robert Brooks, Professeur au Technion (Haifa), est décédé subitement le 5 septembre 2002. Ses travaux sont depuis longtemps au cœur de bien des développements en analyse et géométrie spectrale, et il avait été l'un des participants les plus présents et les plus actifs du trimestre.

Les organisateurs du trimestre

Report on the trimestre "Heat Kernels, Random Walks, and Analysis on Manifolds and Graphs"[1]

"If it's Tuesday, this must be Belgium."

This is the name of a film about a group of tourists who were going from city to city a little too fast. Fortunately in this trimestre there was more time, and while the activities were numerous and extensive, one had the opportunity to delve into various topics in some detail.

To give a part of the mathematical setting, let us review a few classical matters related to calculus and partial differential equations. Fix a positive integer n, and let $\mathbf{R}^n$ be the usual n-dimensional Euclidean space, consisting of n-tuples of real numbers. If $f(x)$ is a real-valued function on $\mathbf{R}^n$ which is twice-continuously differentiable, say, then the *Laplacian* of f is denoted Δf and defined by

$$\Delta f = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f. \tag{1}$$

Let $f_1(x)$, $f_2(x)$ be two real-valued functions on $\mathbf{R}^n$ which are continuous and have compact support, so that they are both equal to 0 outside of a bounded set. More generally, one can assume that f_1, f_2 satisfy suitable decay conditions, etc. The standard inner product of such functions is defined by

$$\langle f_1, f_2 \rangle = \int_{\mathbf{R}^n} f_1(x)\, f_2(x)\, dx. \tag{2}$$

There is another symmetric bilinear form which is closely related to the Laplacian, given by

$$\mathcal{E}(f_1, f_2) = \frac{1}{2} \int_{\mathbf{R}^n} \nabla f_1(x) \cdot \nabla f_2(x)\, dx \tag{3}$$

when f_1, f_2 are continuously differentiable, or satisfy other appropriate regularity conditions. Here $\nabla f(x)$ denotes the gradient of f at x, i.e.,

[1] Some historical notes, mentioned by a colleague: Émile Borel spoke at the opening of the author's home institution, Rice University (originally the Rice Institute) in Houston, Texas, in 1912. Borel published "Molecular theories and mathematics" in connection with his lectures in the Rice Institute Pamphlet, Volume I (1915), 163–193. Henri Poincaré was also invited by President Edgar Odell Lovett and accepted, conditioned on the state of his health, but eventually declined the invitation and subsequently passed away. Borel's paper begins with a tribute to Poincaré, and relates a discussion they had about the trip. Borel indicates that he would have changed his subject to an appreciation of Poincaré's work, except that Vito Volterra was doing exactly that. Volterra's paper appears in the same issue of the Rice Institute Pamphlet, "Henri Poincaré", pp. 133–162. Jacques Hadamard contributed "The early scientific work of Henri Poincaré" and "The later scientific work of Henri Poincaré" to the Rice Institute Pamphlet, Volume IX (1922), 111-183 and Volume XX (1933), 1–86. Hadamard makes the point in the introduction to the first paper that uses for Poincaré's work seemed to take 25 years to be found.

the vector with components $(\partial/\partial x_j)f(x)$, and $v \cdot w$ is the usual inner product on $\mathbf{R}^n$, so that $v \cdot w = \sum_{j=1}^n v_j\, w_j$. If in addition f_1 is twice continuously-differentiable, then

$$\mathcal{E}(f_1, f_2) = -\frac{1}{2}\int_{\mathbf{R}^n} \Delta f_1(x)\, f_2(x)\, dx. \tag{4}$$

This follows from integration by parts.

The *energy* $\mathcal{E}(f)$ of a function f is defined by

$$\mathcal{E}(f) = \mathcal{E}(f, f) = \frac{1}{2}\int_{\mathbf{R}^n} |\nabla f(x)|^2\, dx, \tag{5}$$

where $|v|$ denotes the standard Euclidean length of v, which is the same as saying that $|v|^2 = v \cdot v$. If $\eta(x)$ is another function on $\mathbf{R}^n$, then

$$\frac{d}{ds}\mathcal{E}(f + s\,\eta)\Big|_{s=0} = -\int_{\mathbf{R}^n} \Delta f(x)\, \eta(x)\, dx, \tag{6}$$

under suitable conditions on f and η. This is commonly rephrased as saying that the gradient of the energy functional $\mathcal{E}(f)$ is given by $-\Delta f$, where this statement implicitly uses the inner product (2) on functions on $\mathbf{R}^n$.

A function $u(x, t)$ on $\mathbf{R}^n \times (0, \infty)$ which is twice-continuously differentiable in x and continuously differentiable in x and t is said to satisfy the *heat equation* if

$$\frac{\partial}{\partial t} u = \Delta u. \tag{7}$$

Under modest growth conditions on a function $f(x)$ on $\mathbf{R}^n$, there is a unique continuous function $u(x, t)$ on $\mathbf{R}^n \times [0, \infty)$ such that $u(x, 0) = f(x)$, $u(x, t)$ is infinitely differentiable in x and t when $t > 0$, $u(x, t)$ satisfies the heat equation on $\mathbf{R}^n \times (0, \infty)$, and $u(x, t)$ also satisfies modest growth conditions (which can be related to those of f).

One way to look at the heat equation is as an ordinary differential equation in t, acting in vector spaces of functions of x. To find $u(x, t)$ given $f(x)$ as in the preceding paragraph, one might write

$$u(x, t) = (\exp(t\Delta) f)(x). \tag{8}$$

In fact the Fourier transform gives a useful way to make sense of this.

Aspects of symmetry

Versions of these notions come up in a variety of situations, and a number of these were discussed in the trimestre. In the spirit of the book "Introduction to Fourier Analysis on Euclidean Spaces" by E. Stein and G. Weiss, which also provides a lot of helpful background information for these topics, one might start by considering the symmetries of the objects just described. They are all invariant under translations, and under rotations on $\mathbf{R}^n$. They also behave nicely with respect to dilations on $\mathbf{R}^n$, which is to say under transformations of the form $x \mapsto ax$, where a is a positive real number. In the case of the heat equation, one should use the dilations $(x,t) \mapsto (ax, a^2 t)$, to adjust for the fact that there is one derivative in t and derivatives of order 2 in x.

Instead of Euclidean spaces a basic setting is that of irreducible symmetric spaces of noncompact type, which was discussed in the course of J.-P. Anker. For these one again has translation invariance and forms of rotation invariance, but no dilation invariance. There are counterparts of Fourier analysis here too, for analyzing solutions to the heat equation, but this has some weaknesses differing from the Euclidean case.

In the Euclidean case the solution $u(x,t)$ to the heat equation with initial data $f(x)$ can be expressed in the form

$$u(x,t) = \int_{\mathbf{R}^n} k_t(x-y)\, f(y)\, dy \tag{9}$$

for a function $k_t(x)$ called the *heat kernel*. The fact that the solution can be written in this manner, instead of

$$u(x,t) = \int_{\mathbf{R}^n} k_t(x,y)\, f(y)\, dy, \tag{10}$$

reflects the translation-invariance of the problem in x. The rotation-invariance of the problem implies in turn that $k_t(x)$ is a radial function of x, so that $k_t(x)$ can be written as $h_t(|x|)$ for a function $h_t(r)$ with $t \in (0,\infty)$ and $r \in [0,\infty)$. One can go further and use dilation-invariance to obtain that $k_t(x)$ is of the form $t^{-n/2} h(|x|/\sqrt{t})$ for a function $h(r)$, $r \in [0,\infty)$. It is a classical result, which is a good exercise to derive, that $k_t(x)$ is in fact a Gaussian function of x. This can be viewed in terms of the Fourier transform, or by working out an ordinary differential equation for the function $h(r)$.

In the context of symmetric spaces one can start with a general form for $u(x,t)$ as in (10), and use translation-invariance to reduce to something more like (9). The counterpart of rotation-invariance permits one to reduce the number of variables further, but not in general to 2 variables. Fourier analysis leads to interesting representations for the heat kernel, but fundamental features concerning size and localization are not always so clear from this representation.

Now let us go in a different direction and suppose that we are working on $\mathbf{R}^n$ again, but with a differential operator L with variable coefficients in place of the Laplacian. Specifically, we assume that L is of the form

$$L = \sum_{j,m=1}^{n} \frac{\partial}{\partial x_j}\, a_{j,m}(x)\, \frac{\partial}{\partial x_m}, \tag{11}$$

where $a_{j,m}(x)$ are bounded real-valued functions which satisfy

$$a_{j,m}(x) = a_{m,j}(x) \tag{12}$$

and

$$|v|^2 \leq \sum_{j,m=1}^{n} a_{j,m}(x)\, v_j\, v_m \tag{13}$$

for all $v \in \mathbf{R}^n$. In other words, $(a_{j,m}(x))_{j,m}$ are positive-definite real symmetric matrices which are uniformly bounded in x and bounded from below in the sense of matrices by the identity matrix. Because the coefficients are allowed to depend on x, we lose in general the invariance under translations, rotations, or dilations, and the heat kernel should be written as $k_t(x, y)$, with $x, y \in \mathbf{R}^n$ and $t > 0$, as in (10). However, there are vestiges of these invariances, in that translations and rotations of L lead to operators of the same type, and similarly for dilations if one includes suitable scale-factors. While the precise form of the heat kernel may not be easy to describe, one can try to show that it has many properties in common with the Gaussian kernels in the case of the standard Laplacian.

One can go further and consider coefficients $a_{j,m}(x)$ which are not symmetric in j or m, and perhaps not even real-valued. For the latter one can adjust (13) by taking the real part of the right side, so that one still has "uniform ellipticity". More generally one can allow operators of order larger than 2, and vector-valued functions and systems of differential equations. Questions related to these situations were discussed in the courses of P. Auscher and P. Tchamitchian, and of S. Hofmann and A. McIntosh.

Note that it still makes sense to talk about

$$\exp(tL) \tag{14}$$

in this type of situation, using spectral theory. This works more nicely when the coefficients $a_{j,m}(x)$ are real and symmetric, so that the operator L is self-adjoint (with a suitable choice of domain). Even without these conditions, one can define (14), using resolvent integrals. For that matter, one can define more general functions of L, and part of the interest of the heat kernels is that the exponentials (14) and related operators can make good building blocks for studying other functions of L.

On a connected Lie group H one can again look at second-order elliptic differential operators L which are invariant under translations, but in general H can be noncommutative and one should be careful to specify whether L is invariant under left translations, right translations, or both. In the case of Lie groups which are nilpotent, such as the Heisenberg groups, dilations can be used in much the same manner as on Euclidean spaces to have an extra degree of symmetry. In the course of W. Hebisch, solvable Lie groups and operators on them were treated, for which there is a delicate interplay between exponential growth on the one hand and having a fair amount of commutativity around on the other hand.

S. Lang gave a series of lectures concerning deep questions of expansions for heat kernels on the locally symmetric spaces (of finite volume)

$$SL(n,\mathbf{R})/SL(n,\mathbf{Z}), \quad SL(n,\mathbf{C})/SL(n,\mathbf{Z}[i]), \tag{15}$$

where $\mathbf{Z}$ denotes the set of integers, and $\mathbf{Z}[i]$ is the set of complex numbers whose real and imaginary parts are integers.

Discrete settings

Let us consider $\mathbf{Z}^n$ now instead of $\mathbf{R}^n$. If x, y are elements of $\mathbf{Z}^n$, let us say that x and y are *adjacent* if $|x-y|=1$. Thus x and y are adjacent if they agree in all but one component, where they differ by ± 1. If $f(x)$ is a function on $\mathbf{Z}^n$, define $A(f)$ on $\mathbf{Z}^n$ by

$$A(f)(x) = \frac{1}{2n} \sum_{\substack{y\in\mathbf{Z}^n \\ |x-y|=1}} f(y), \tag{16}$$

so that $A(f)(x)$ is the average of f over the $2n$ elements of $\mathbf{Z}^n$ adjacent to x.

The linear operator $A-I$ on functions on $\mathbf{Z}^n$, where I denotes the identity operator, is a discrete version of the Laplacian. This makes more sense if one writes the classical Laplacian of a twice continuously-differentiable function h at a point x as

$$\Delta(h)(x) = \lim_{r\to 0} \frac{1}{r^2}(\mathrm{Av}(h)(x,r) - h(x)), \tag{17}$$

with $\mathrm{Av}(h)(x,r)$ equal to the average of h over the sphere with center x and radius r.

The analogue of the heat equation for a function $u(x,t)$ with x in $\mathbf{Z}^n$ and t ranging through nonnegative integers can be written as

$$u(x,t+1) = \frac{1}{2n} \sum_{\substack{y\in\mathbf{Z}^n \\ |x-y|=1}} u(x,t), \tag{18}$$

which is the same as saying that $u(x,t+1)$ is given by applying the operator A to $u(x,t)$ as a function of x. To make this look more like the classical heat equation, one can reexpress this as saying that $u(x, t+1) - u(x, t)$, which is like the "derivative" of u in t, is equal to $A - I$ applied to $u(x, t)$ as a function of x. Clearly, for any function $f(x)$ on $\mathbf{Z}^n$, there is a unique function $u(x, t)$ defined for x in $\mathbf{Z}^n$ and t a nonnegative integer such that $u(x, 0) = f(x)$ for all x in $\mathbf{Z}^n$ and $u(x, t)$ satisfies the heat equation above for all x and t. In fact, $u(x, t)$ can be written as

$$u(x, t) = (A^t)(f)(x), \tag{19}$$

in analogy with (8).

In analogy with (9), we can write

$$u(x, t) = \sum_{y \in \mathbf{Z}^n} p_t(x - y)\, f(y), \tag{20}$$

where the "heat kernel" $p_t(w)$ is defined for t a nonnegative integer and w in $\mathbf{Z}^n$. Specifically, $p_0(w)$ is equal to 0 when $w \neq 0$ and to 1 when $w = 0$, $p_1(w)$ is equal to 0 when w is not adjacent to 0 and to $1/(2n)$ when w is adjacent to 0, and $p_t(w)$ can easily be determined explicitly.

In fact, $p_t(x - y)$ is the probability that the standard random walk on $\mathbf{Z}^n$ goes from x to y in exactly n steps. In the continuous setting there are similar statements for Brownian motion and other processes associated to second-order differential operators.

That the heat kernel in (20) is of the form $p_t(x - y)$, rather than $p_t(x, y)$, reflects the translation-invariance here, just as in the classical case on $\mathbf{R}^n$. Of course one can consider other graphs instead of $\mathbf{Z}^n$, with similar objects as defined above, and with a formula of the type

$$u(x, t) = \sum p_t(x, y)\, f(y) \tag{21}$$

in place of (20).

The course of T. Sunada dealt with *crystal lattices*, which are characterized in terms of a large abelian group of symmetries. The graphs $\mathbf{Z}^n$ are a very special case of this, and numerous other configurations are possible. In W. Woess' course, techniques of *generating functions* were discussed, which can lead to remarkable formulas and information about random walks. Part of M. Barlow's course was concerned with random walks on graphs with self-similarity, and the effect of self-similarity on the heat kernel.

In analogy with second-order differential operators on $\mathbf{R}^n$ with variable coefficients, one can consider random walks and discrete Laplacians on $\mathbf{Z}^n$ in which the weighting factors vary from point to point. One does not need to stick to $\mathbf{R}^n$ or $\mathbf{Z}^n$ here; one can work on manifolds or graphs, or more generally metric spaces equipped with a measure. Several of the courses dealt with different facets of this, including Sobolev spaces and Sobolev or Poincaré inequalities.

R. Brooks discussed in his course Riemann surfaces, graphs, correspondences between them, and lower bounds for positive eigenvalues for the Laplacian for both.

Additional topics

Let p be a real number, $p > 1$. For suitable functions $f(x)$ on $\mathbf{R}^n$, consider the *p-energy functional*

$$\mathcal{E}_p(f) = \frac{1}{p} \int_{\mathbf{R}^n} |\nabla f(x)|^p. \tag{22}$$

This is the same as $\mathcal{E}(f)$ in (5) when $p = 2$, but there is not a bilinear version as in (3) when $p \neq 2$. However, one can again consider the derivative of $\mathcal{E}_p(f)$ in f for all p, and this leads to a nonlinear (when $p \neq 2$) second-order differential operator known as the *p-Laplacian.*

The p-energy is invariant under translations and rotations, and scales under dilations in a simple way, just as when $p = 2$. For $p = n$ there is additional symmetry, known as *conformal invariance.*

One can consider more complicated functionals which behave in roughly the same manner in terms of size, but which incorporate "variable coefficients" into the picture. When $p = n$ there is a "quasi-invariance" of the energy under *quasiregular* mappings, which are defined in terms of a pointwise quasiconformality property (where the nth power of the norm of the differential of the mapping is bounded by a constant times the Jacobian, i.e., the determinant of the differential of the mapping). Quasiregular mappings, unlike quasiconformal mappings, are allowed to have branching, analogous to holomorphic mappings in the complex plane which are not one-to-one. The *quasi-invariance* of the p-energy when $p = n$ states that the energy functional is transformed by a quasiregular change of variables into an energy functional of roughly the same type, but with variable coefficients which satisfy bounds in terms of the quasiregularity constant. As a result, a solution of the n-Laplace equation is transformed, after composition with a quasiregular mapping, into a solution of an analogous equation with variable coefficients, still with suitable boundedness and ellipticity conditions. This is an important tool in the study of quasiregular mappings, as discussed in the course of I. Holopainen.

Even with the extra nonlinearity, there are similar issues concerning the relationship between the geometry of a space and the behavior of solutions of differential equations or inequalities as before.

A different kind of nonlinearity was treated in the course of K.-T. Sturm, with averages, heat flows, and random processes taking values in a metric space, under general conditions of nonpositive curvature. It can be clear how to take a weighted average of two points in a metric space, using a point along a geodesic arc that joins them, but for more than two points not lying on the same geodesic the situation becomes more complicated. A fascinating feature of the probabilistic point of view is that in a sequence of independent samples one can use the ordering of the sequence to apply the two-point case step-by-step; it turns out that there are results to the effect that the limit of this exists and is the same almost surely, and that the common answer is the same as one produced from another procedure which deals with all points in the average at the same time.

The courses of B. Driver and L. Saloff-Coste were concerned with analysis on infinite-dimensional spaces. Specifically, Driver's course dealt with Wiener space, spaces of paths in manifolds, and loop groups, while Saloff-Coste's course addressed locally-compact and connected topological groups, such as infinite products of finite-dimensional compact connected Lie groups.

Of course the brief overview given here is not at all intended to be exhaustive. Fortunately, a volume is in preparation containing surveys and other material from the trimestre, in which much more information can be found.

Some topics concerning analysis on metric spaces and semigroups of operators

Classical analysis on Euclidean spaces (as in** [100, 103]**)

Fix a positive integer n, and consider the Euclidean space $\mathbf{R}^n$ equipped with the standard distance function $|x-y|$ and Lebesgue measure. If $f(x)$ is a locally-integrable function on $\mathbf{R}^n$, then the *Hardy–Littlewood maximal function* $f^*(x)$ associated to f is defined by

$$f^*(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y), \tag{23}$$

where the supremum is taken over all open balls B in $\mathbf{R}^n$ which contain x. The supremum may be $+\infty$, so that f^* is actually a (nonnegative) extended real-valued function. This is sometimes referred to as the *uncentered* maximal function, and there are variants defined in terms of balls centered at x, or using cubes instead of balls. A nice feature of $f^*(x)$ is that it is lower semicontinuous, which is to say that

$$\{x \in \mathbf{R}^n : f^*(x) > t\} \tag{24}$$

is an open subset of $\mathbf{R}^n$ for each positive real number t. Indeed, if $f^*(x) > t$ for some x, then there is a ball B containing x such that

$$\frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y) > t, \tag{25}$$

and it follows that $f^*(z) > t$ for all z in B.

Clearly the supremum of f^* is less than or equal to the L^∞ norm of f. A famous *weak type* $(1,1)$ result says that

$$|\{x \in M : f^*(x) > t\}| \leq C(n)\, t^{-1} \|f\|_1, \tag{26}$$

for some constant $C(n) > 0$ that depends only on the dimension n and all functions f, where $|E|$ denotes the Lebesgue measure of a set E and $\|f\|_1$ is the usual L^1 norm of f. In particular, f^* is finite almost everywhere in this case. For $p > 1$ there is a *strong type* result, which means that

$$\|f^*\|_p \leq C(n,p)\, \|f\|_p \tag{27}$$

for some constant $C(n,p)$ which depends only on n and p, and where $\|f\|_p$ denotes the usual L^p norm of f. This can in fact be derived from the preceding estimates for $p = 1, \infty$ through a general interpolation result.

One might be interested in other kinds of averages of f, such as those given by integrating f against the Poisson kernel or the Gauss–Weierstrass kernel. These are exactly the quantities which arise in the extensions of f to the upper half space $\mathbf{R}^n \times (0,\infty)$ which are harmonic or satisfy the heat equation (and which satisfy additional mild growth conditions to avoid modest ambiguities). Fortunately, these averages can be estimated in terms of averages over balls in a simple way, so that the corresponding maximal functions are bounded in terms of f^*. Thus the inequalities above for f^* provide basic results about the boundary behavior of solutions to the Laplace and heat equations on $\mathbf{R}^n \times (0,\infty)$.

Some other interesting operators are the singular integral operators

$$R_j(f)(x) = p.v. \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}}\, f(y)\, dy, \tag{28}$$

$1 \leq j \leq n$, and

$$I_{it}(f)(x) = p.v. \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n+it}}\, f(y)\, dy, \tag{29}$$

$t \in \mathbf{R}$, $t \neq 0$. Some care is involved in taking the principal values, especially in the second case. For I_{it}, different ways of defining the principal values will even lead to different answers, but the difference is rather mild (a multiple of the identity operator).

One can show that these operators are bounded on L^2 using special structure related to $p = 2$, i.e., Fourier transform and Hilbert space methods. This can be extended to boundedness on L^p when $1 < p < \infty$ and the weak type $(1,1)$ property for $p = 1$ using well-known techniques in harmonic analysis. For $p = \infty$ there are estimates in terms of BMO, as a substitute for L^∞ bounds which do not work. Similar results apply to numerous other operators of similar type.

Spaces of homogeneous type [20, 21]

A *space of homogeneous type* can be described as a triple $(M, d(x,y), \mu)$, where M is a nonempty set, $d(x,y)$ is a metric on M (and thus is a symmetric nonnegative real-valued function on $M \times M$ which vanishes exactly when $x = y$ and satisfies the triangle inequality), and μ is a *doubling measure* on M. The latter means that μ is a nonnegative Borel measure which assigns positive finite measure to open balls in M, and for which there is a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r)) \tag{30}$$

for every ball $B(x,r)$ in M. Of course (30) implies that μ assigns positive finite measure to every open ball in M as soon as this holds for a single such ball. One might also ask that M be complete, in the sense that Cauchy sequences converge, or that open subsets of M be realizable as countable unions of compact sets. Basic examples of spaces of homogeneous type are given by Euclidean spaces with the standard metric and Lebesgue measure. Reasonably-smooth domains or manifolds are also included in this notion.

It can be convenient to allow $d(x,y)$ to be a *quasimetric* instead of a metric, which means that a positive constant factor is allowed on the right side of the triangle inequality, and the notion of a space of homogeneous type is often formulated in this manner. As in [79], there are always metrics not too far from quasimetrics, so that for many purposes one might as well restrict to metrics.

The Hardy–Littlewood maximal function f^* associated to a locally-integrable function f can be defined on a space of homogeneous type in the same manner as on Euclidean spaces. A basic fact is that the weak type $(1,1)$ estimate extends to this general setting. The supremum of f^* is still bounded by the L^∞ norm of f, and L^p estimates for $1 < p < \infty$ follow from the $p = 1, \infty$ estimates through general interpolation arguments, as before.

Another basic result is that one has "Calderón–Zygmund inequalities" for singular integral operators analogous to those on $\mathbf{R}^n$. That is, one can start with a linear operator T which is bounded on L^2, or some other fixed L^{p_1}, and which is associated to a kernel that satisfies suitable size and smoothness conditions, and derive boundedness on L^p for all $1 < p < \infty$ and a weak-type inequality for $p = 1$. One can also get BMO extimates for $p = \infty$, estimates on Hardy spaces as an alternative to the weak-type inequality for $p = 1$ as well as allowing for some $p < 1$, etc. The compatibility between the metric and the measure given by the doubling condition is quite remarkable.

Let us mention two classes of examples of spaces of homogeneous type which were examined on their own before the general notion. In the first case, which was studied by my colleague Frank Jones [65], one takes $\mathbf{R}^n \times \mathbf{R}$ with the distance between two points (x, s), (y, t) defined to be

$$|x - y| + |s - t|^{1/2}, \tag{31}$$

where $|x - y|$, $|s - t|$ denote the usual distances in $\mathbf{R}^n$, $\mathbf{R}$, respectively. Sometimes other expressions are used for essentially the same geometry; a key point is that the distance behaves well under the non-isotropic dilations

$$(x, t) \mapsto (rx, r^2 t), \tag{32}$$

for $r > 0$, just as the ordinary metric on $\mathbf{R}^n$ behaves well under the dilations $x \mapsto rx$. Of course the metric is also invariant under translations on $\mathbf{R}^n \times \mathbf{R}$, and is compatible with the usual topology. For the measure one still uses Lebesgue measure. The measure of a ball of radius ρ is a constant multiple of ρ^{n+2}, and the doubling condition is satisfied. In this case the singular integral theory can be applied to operators related to the heat operator, whereas the standard geometry on $\mathbf{R}^n$ fits with operators related to the Laplacian. Note that there is a kind of tricky point here, in which the t parameter is included in the underlying space.

A second basic situation corresponds to the unit sphere in $\mathbf{C}^n$, which, for $n \geq 2$, has a non-Euclidean geometry which is adapted to several complex variables, holomorphic functions on the unit ball in $\mathbf{C}^n$, etc. Just as in the previous case, one can still use ordinary Lebesgue measure on the sphere, and this measure is doubling with respect to the non-Euclidean geometry. (For that matter, it is also doubling with respect to the usual Euclidean geometry.) The Hardy–Littlewood maximal function with respect to the non-Euclidean geometry is closely connected to maximal functions and limits for holomorphic functions in the ball along certain "admissible" regions, just as the classical maximal function is connected to nontangential maximal functions for holomorphic functions in one complex variable or harmonic functions in several real variables. A fundamental singular integral operator in this situation is the *Szegö projection*, which is the orthogonal projection from L^2 of the unit sphere onto the subspace of functions which are boundary values of holomorphic functions on the ball. This operator is bounded on L^2 with norm 1 by definition, and its kernel can be computed explicitly. With respect to the non-Euclidean geometry, the kernel satisfies the appropriate size and smoothness conditions, so that the operator is in fact bounded on L^p, $1 < p < \infty$, and so on. See [68, 69, 70, 71, 72, 74, 85, 101, 102].

Semigroups of operators

In another direction, suppose that $\mathcal{B}$ is a Banach space, and that $\{T_t\}_{t \ge 0}$ is a *semigroup of bounded operators* on $\mathcal{B}$. Specifically, assume that T_0 is the identity operator I, that the operator norm of T_t is bounded by some constant k for $0 \le t \le 1$, that

$$T_{s+t} = T_s \circ T_t \tag{33}$$

for all $s, t \ge 0$, and that $\lim_{t \to 0} T_t(f) = f$ for all f in $\mathcal{B}$. Of course the semigroup property together with the uniform bound for the operator norm of the T_t's for $0 \le t \le 1$ implies an exponentially-increasing bound for the operator norm of T_t for all t's.

There is a remarkable amount of mathematics around this kind of situation. In fact, this is just the beginning; one can add relatively-simple hypotheses which occur in numerous settings and which add quite a bit more structure. As a basic distinction, one might think of $\{T_t\}_{t \ge 0}$ as being a semigroup of unitary transformations on a Hilbert space, or a semigroup of invertible linear mappings on a Banach space more generally, as is associated to solutions of a wave equation, or one might think of $\{T_t\}_{t \ge 0}$ as defining a diffusion, as is associated to solutions of a heat equation.

Here we shall mostly focus on the second type of situation. We assume now that we have a measure space M with a positive measure μ, and we take for our Banach space $\mathcal{B}$ the Hilbert space $L^2(M, \mu)$. We ask too that each T_t be self-adjoint and *positivity-preserving*, which means that for each nonnegative function f on M, $T_t(t)$ is also a nonnegative function on M for every $t \ge 0$. Each of these conditions is significant in its own right, and part of the beauty of the subject arises from the interplay between them.

Let us also ask that the T_t's extend to bounded operators on $L^p(M, \mu)$ for each $1 \le p \le \infty$, and in fact that the T_t's are *contractions* on all L^p, which is to say that the operator norms are all less than or equal to 1. If $\mathbf{1}$ denotes the function on M which is identically equal to 1, then we ask that $T_t(\mathbf{1}) = \mathbf{1}$ for all t. These conditions are satisfied by the semigroups associated to the heat kernel and Poisson kernel on $\mathbf{R}^n$, for instance.

A famous result of Stein states that the maximal function inequalities

$$\| \sup_{t > 0} |T_t(f)| \|_p \le A_p \, \|f\|_p \tag{34}$$

hold for $1 < p \le \infty$, i.e., for some constant A_p and all functions f in L^p. In this setting there is also a "singular integral operator" theory, for operators which are functions of the generator of the semigroup. Boundedness on L^2 for these operators is easily determined through the spectral representation. Some general conditions for boundedness on L^p

are described in [99], and of course more precise information depends on the particular situation.

The significance of self-adjointness is illustrated by the example where T_t is defined on functions on $\mathbf{R}$ to be translation by t. This semigroup of operators is positivity-preserving and preserves all L^p norms, but the maximal inequality fails completely for $p < \infty$. In this case it is natural to consider averages of $T_t f$ and suprema of the averages, as in ergodic theory, and as for semigroups associated to measure-preserving transformations on the underlying measure space more generally.

Semigroups and geometry

There are very interesting combinations of the spaces of homogeneous type and semigroups of operators pictures, involving bounds for kernels of semigroups, and L^p mapping properties of operators related to the semigroup. See [3, 4, 25, 26, 27, 34, 35], for instance. Another perspective has recently been studied in [48], with the following set-up. One assumes again that $(M, d(x, y))$ is a metric space, that μ is a positive Borel measure on it, and that T_t is a symmetric contraction semigroup of linear operators as before. Now one asks in addition that for $t > 0$ the operator T_t is defined by a nonnegative kernel $k_t(x, y)$, so that

$$T_t(f)(x) = \int_M k_t(x, y)\, f(y)\, d\mu(y). \tag{35}$$

For the kernel $k_t(x, y)$ one considers upper and lower bounds of the form

$$\frac{1}{t^{\alpha/\beta}}\, \varphi_1\left(\frac{d(x, y)^\beta}{t}\right) \le k_t(x, y) \le \frac{1}{t^{\alpha/\beta}}\, \varphi_2\left(\frac{d(x, y)^\beta}{t}\right). \tag{36}$$

Here α, β are positive constants, and $\varphi_1(u)$, $\varphi_2(u)$ are monotone decreasing positive functions on $[0, \infty)$, with $\varphi_1(u_1) > 0$ for some $u_1 > 0$ and $\varphi_2(u)$ normally asked to satisfy decay conditions.

The parameter α is related to volume growth in M, and this is discussed in [48]. The connection between β and the geometry of M is also treated in [48]. For the standard heat semigroup on Euclidean spaces, β is always equal to 2. There are heat semigroups associated to subelliptic operators in place of the ordinary Laplacian which also satisfy these conditions with $\beta = 2$, with respect to an associated metric. A basic version of this arises for the unit sphere in $\mathbf{C}^n$, $n \ge 2$, and non-Euclidean geometry on it, as indicated earlier. There are a number of fractals such as Sierpinski gaskets and carpets and semigroups on them which satisfy the conditions above with various values of β. Compare with [6, 7, 8, 9, 10, 37, 67, 75].

Without the semigroup property, there are well-known fairly simple constructions of approximations to the identity on spaces of homogeneous type with nice properties. The semigroup property of course imposes very strong restrictions. For that matter, commutativity of the operators in the family is a substantial condition.

Let us note that decay conditions on $\varphi_2(u)$ above can be quite significant. A very nice contraction semigroup on $\mathbf{R}^n$ is given by the Poisson kernel, and for this kernel the decay is not very fast. Modest decay conditions for the kernel are adequate for a number of applications, even if they are not sufficient for other results, as in [48].

Analysis on fractals like Sierpinski gaskets and carpets

Of course decay conditions for the kernel of a semigroup are closely connected to locality conditions for the generator of the semigroup. In the classical cases on Euclidean spaces or tori for periodic functions, etc., the heat kernel has fast decay and is generated by the Laplace operator, while the Poisson kernel does not have very fast decay and is generated by a constant multiple of the *square root* of the Laplace operator, which is not a local operator.

The fast decay for the kernels of the semigroups on Sierpinski gaskets and carpets mentioned before reflects the fact that the generators are nice local operators, versions of "Laplacians" for these fractals. The fractal structures play a role and are reflected in the parameter β, but still there are nice operators which are like differential operators.

For example, there are remarkable results concerning elliptic and parabolic Harnack inequalities for these operators. See [6, 7, 8, 9, 67].

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References

[1] G. Alexopoulos, *Sub-Laplacians with Drift on Lie Groups of Polynomial Volume Growth*, Memoirs of the American Mathematical Society **739**, 2002.
[2] L. Ambrosio and P. Tilli, *Selected Topics on "Analysis in Metric Spaces"*, Scuola Normale Superiore, Pisa, 2000.
[3] P. Auscher, A. McIntosh, and P. Tchamitchian, *Heat kernels for second order complex elliptic operators and applications*, Journal of Functional Analysis **152** (1998), 22–73.
[4] P. Auscher and P. Tchamitchian, *Square root problem for divergence operators and related topics*, Astérisque **249**, 1998.
[5] D. Bakry, T. Coulhon, M. Ledoux, and L. Saloff-Coste, *Sobolev inequalities in disguise*, Indiana University Mathematics Journal **44** (1995), 1033–1074.
[6] M. Barlow and R. Bass, *Transition densities for Brownian motion on the Sierpinski carpet*, Probability Theory and Related Fields **91** (1992), 307–330.
[7] M. Barlow and R. Bass, *Coupling and Harnack inequalities for Sierpinski carpets*, Bulletin of the American Mathematical Society (N.S.) **29** (1993), 208–212.
[8] M. Barlow and R. Bass, *Brownian motion and harmonic analysis on Sierpinski carpets*, Canadian Journal of Mathematics **51** (1999), 673–744.

[9] M. Barlow and R. Bass, *Divergence form operators on fractal-like domains*, Journal of Functional Analysis **175** (2000), 214–247.

[10] M. Barlow and E. Perkins, *Brownian motion on the Sierpinski gasket*, Probability Theory and Related Fields **79** (1988), 542–624.

[11] R. Beals, B. Gaveau, and P. Greiner, *Hamilton–Jacobi theory and the heat kernel on Heisenberg groups*, Journal de Mathématiques Pures et Appliquées (9) **79** (2000), 633–689.

[12] R. Beals, P. Greiner, and N. Stanton, *The heat equation and geometry of CR manifolds*, Bulletin of the American Mathematical Society (N.S.) **10** (1984), 275–276.

[13] R. Beals, P. Greiner, and N. Stanton, *The heat equation on a CR manifold*, Journal of Differential Geometry **20** (1984), 343–387.

[14] A. Bellow, *Transference principles in ergodic theory*, in *Harmonic Analysis and Partial Differential Equations: Essays in Honor of Alberto Calderón* (M. Christ, C. Kenig, and C. Sadosky, eds.), 27–39, University of Chicago Press, Chicago, 1999.

[15] A. Bellow and A. Calderón, *A weak-type inequality for convolution products*, in *Harmonic Analysis and Partial Differential Equations: Essays in Honor of Alberto Calderón* (M. Christ, C. Kenig, and C. Sadosky, eds.), 41–48, University of Chicago Press, Chicago, 1999.

[16] E. Berkson, M. Paluszynski, and G. Weiss, *Transference couples and weighted maximal estimates*, in *Harmonic Analysis and Partial Differential Equations: Essays in Honor of Alberto Calderón* (M. Christ, C. Kenig, and C. Sadosky, eds.), 49–67, University of Chicago Press, Chicago, 1999.

[17] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geometric and Functional Analysis **7** (1997), 245–268.

[18] M. Bourdon and H. Pajot, *Poincaré inequalities and quasiconformal structures on the boundary of some hyperbolic buildings*, Proceedings of the American Mathematical Society **127** (1999), 2315–2324.

[19] M. Bourdon and H. Pajot, *Cohomologie ℓ_p et espaces de Besov*, preprint, 2002.

[20] R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Mathematics **242**, Springer-Verlag, 1971.

[21] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bulletin of the American Mathematical Society **83** (1977), 569–645.

[22] R. Coifman and G. Weiss, *Transference Methods in Analysis*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics **31**, American Mathematical Society, 1977.

[23] T. Coulhon, *Espaces de Lipschitz et inégalités de Poincaré*, Journal of Functional Analysis **136** (1996), 81–113.

[24] T. Coulhon, *Ultracontractivity and Nash type inequalities*, Journal of Functional Analysis **141** (1996), 510–539.

[25] T. Coulhon and X. Duong, *Riesz transforms for $1 \leq p \leq 2$*, Transactions of the American Mathematical Society **351** (1999), 1151–1169.

[26] T. Coulhon and X. Duong, *Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss*, Advances in Differential Equations **5** (2000), 343–368.

[27] T. Coulhon and X. Duong, *Riesz transforms for $p > 2$*, Comptes Rendus de l'Académie des Sciences Paris Sér. I Math. **332** (2001), 975–980.

[28] T. Coulhon and L. Saloff-Coste, *Isopérimétrie pour les groupes et les variétés*, Revista Matemática Iberoamericana **9** (1993), 293–314.

[29] D. Danielli, N. Garofalo, and D.-M. Nhieu, *Non-doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot–Carathéodory spaces*, preprint, 2002.

[30] G. David and S. Semmes, *Strong A_∞-weights, Sobolev inequalities, and quasiconformal mappings*, in *Analysis and Partial Differential Equations* (C. Sadosky, ed.), 101–111, Marcel Dekker, 1990.

[31] G. David and S. Semmes, *Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure*, Oxford University Press, 1997.

[32] R. DeVore and R. Sharpley, *Maximal Functions Measuring Smoothness*, Memoirs of the American Mathematical Society **293**, 1984.

[33] J. Duoandikoetxea, *Fourier Analysis*, translated and revised by D. Cruz-Uribe, SFO, American Mathematical Society, 2001.
[34] X. Duong and A. McIntosh, *Singular integral operators with non-smooth kernels on irregular domains*, Revista Matemática Iberoamericana **15** (1999), 233–265; corrigenda, **16** (2000), 217.
[35] X. Duong and D. Robinson, *Semigroup kernels, Poisson bounds, and holomorphic functional calculus*, Journal of Functional Analysis **142** (1996), 89–128.
[36] J. Fernández and J. Rodríguez, *Area growth and Green's function of Riemann surfaces*, Arkiv för Matematik **30** (1992), 83–92.
[37] P. Fitzsimmons, B. Hambly, and T. Kumagai, *Transition density estimates for Brownian motion on affine nested fractals*, Communications in Mathematical Physics **165** (1994), 595–620.
[38] G. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, 1982.
[39] B. Franchi, P. Hajłasz, and P. Koskela, *Definitions of Sobolev classes on metric spaces*, Annales de l'Institut Fourier (Grenoble) **49** (1999), 1903–1924.
[40] B. Franchi, R. Serapioni, and F. Serra Cassano, *Sur les ensembles de périmètre fini dans le groupe de Heisenberg*, Comptes Rendus de l'Académie des Sciences Paris Sér. I Math. **329** (1999), 183–188.
[41] B. Franchi, R. Serapioni, and F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Mathematiche Annalen **321** (2001), 479–531.
[42] B. Franchi, R. Serapioni, and F. Serra Cassano, *Regular hypersurfaces, intrinsic perimeter, and implicit function theorem in Carnot groups*, to appear, Communications in Analysis and Geometry.
[43] M. Fukushima, *Dirichlet forms, diffusion processes and spectral dimension for nested fractals*, in *Ideas and Methods in Mathematical Analysis, Stochastics, and Applications (Oslo, 1988)*, 151–161, Cambridge University Press, 1992.
[44] J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, and J. Torrea, *Functional calculus for the Ornstein–Uhlenbeck operator*, Journal of Functional Analysis **183** (2001), 413–450.
[45] J. García-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.
[46] A. Grigoryan, *The heat equation on noncompact Riemannian manifolds*, Mat. Sb. **182** (1991), 55–87; Math. USSR-Sb. **72** (1992), 47–77.
[47] A. Grigoryan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bulletin of the American Mathematical Society (N.S.) **36** (1999), 135–249.
[48] A. Grigoryan, J. Hu, and K.-S. Lau, *Heat kernels on metric-measure spaces and an application to semi-linear elliptic equations*, preprint, 2002.
[49] A. Grigoryan and A. Telcs, *Sub-Gaussian estimates of heat kernels on infinite graphs*, Duke Mathematical Journal **109** (2001), 451–510.
[50] M. Gromov, *Structures Métriques pour les Variétés Riemanniennes* (J. Lafontaine and P. Pansu, eds.), Cedic/Fernand Nathan, Paris, 1981.
[51] M. Gromov, *Carnot-Carathéodory spaces seen from within*, in *Sub-Riemannian Geometry* (A. Bellaiche and J.-J. Risler, eds.), 79–323, Birkhäuser, 1996,
[52] M. Gromov et al., *Metric Structures for Riemannian and non-Riemannian Spaces*, Birkhäuser, 1999.
[53] P. Hajłasz, *Sobolev spaces on an arbitrary metric space*, Potential Analysis **5** (1996), 403–415.
[54] P. Hajłasz and P. Koskela, *Sobolev meets Poincaré*, Comptes Rendus de l'Académie des Sciences Paris Sér. I Math. **320** (1995), 1211–1215.
[55] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Memoirs of the American Mathematical Society **668**, 2000.
[56] B. Hambly and T. Kumagai, *Transition density estimates for diffusion processes on post critically finite self-similar fractals*, Proceedings of the London Mathematical Society (3) **78** (1999), 431–458.

[57] E. Hebey, *Sobolev Spaces on Riemannian Manifolds*, Lecture Notes in Mathematics **1635**, Springer-Verlag, 1996.

[58] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lectuure Notes in Mathematics **5**, American Mathematical Society and Courant Institute of Mathematical Sciences, 1999.

[59] W. Hebisch, *Spectral multipliers on exponential growth solvable Lie groups*, Mathematische Zeitschrift **229** (1998), 435–441.

[60] W. Hebisch, *Spectral multipliers on metabelian groups*, Revista Matemática Iberoamericana **16** (2000), 597–604.

[61] W. Hebisch and L. Saloff-Coste, *On the relation between elliptic and parabolic Harnack inequalities*, Annales de l'Institut Fourier (Grenoble) **51** (2001), 1437–1481.

[62] J. Heinonen, *Calculus on Carnot groups*, Notes from lectures given at the Jyväskylä Fall School of Analysis in October, 1994), Ber. Jyväskylä Univ. Math. Inst. **68** (1995), 1–31.

[63] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, 2001.

[64] E. Hille and R. Phillips, *Functional Analysis and Semi-Groups*, Colloquium Publications **31**, American Mathematical Society, 1957.

[65] F. Jones, *A class of singular integrals*, American Journal of Mathematics **86** (1964), 441–462.

[66] J.-L. Journé, *Calderón–Zygmund Operators, Pseudodifferential Operators, and the Cauchy Integral of Calderón*, Lecture Notes in Mathematics **994**, Springer-Verlag, 1983.

[67] J. Kigami, *Analysis on Fractals*, Cambridge University Press, 2001.

[68] A. Koranyi, *The Poisson integral for generalized half-planes and bounded symmetric domains*, Annals of Mathematics (2) **82** (1965), 332–350.

[69] A. Koranyi, *Harmonic functions on Hermitian hyperbolic space*, Transactions of the American Mathematical Society **135** (1969), 507–516.

[70] A. Koranyi, *A remark on boundary values of functions of several complex variables*, in *Several Complex Variables, I*, Lecture Notes in Mathematics **155**, 115–121, Springer-Verlag, 1970.

[71] A. Koranyi and S. Vagi, *Intégrales singulières sur certains espaces homogènes*, Comptes Rendus Acad. Sci. Paris Sér. A-B **268** (1969), A765–A768.

[72] A. Koranyi and S. Vagi, *Singular integrals on homogeneous spaces and some problems of classical analysis*, Annali della Scuola Normale Superiore di Pisa (3) **25** (1971), 575–648.

[73] S. Krantz, *A Panorama of Harmonic Analysis*, Carus Mathematical Monographs **27**, Mathematical Association of America, 1999.

[74] S. Krantz, *Function Theory of Several Complex Variables*, American Mathematical Society Chelsea Publishing, 2001.

[75] T. Kumagai, *Estimates of the transition densities for Brownian motion on nested fractals*, Probability Theory and Related Fields **96** (1993), 205–224.

[76] T. Laakso, *Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality*, Geometric and Functional Analysis **10** (2000), 111–123.

[77] T. Laakso, *Plane with A_∞-weighted metric not bilipschitz embeddable to $\mathbf{R}^n$*, Reports of the Department of Mathematics, University of Helsinki **258**, 2000.

[78] G. Leonardi and S. Rigot, *Isoperimetric sets on Carnot groups*, to appear, Houston Journal of Mathematics.

[79] R. Macias and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Advances in Mathematics **33** (1979), 257–270.

[80] R. Macias and C. Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type*, Advances in Mathematics **33** (1979), 271–309.

[81] A. Nahmod, *Geometry of operators and spectral analysis on spaces of homogeneous type*, Comptes Rendus de l'Académie des Sciences Paris, Sér. I Math. **313** (1991), 721–725.

[82] A. Nahmod, *Generalized uncertainty principles on spaces of homogeneous type*, Journal of Functional Analysis **119** (1994), 171–209.

[83] M. Picardello and W. Woess, editors, *Random Walks and Discrete Potential Theory*

(Proceedings of the Conference Held in Cortona, June, 1997), Cambridge University Press, 1999.

[84] D. Robinson, *Elliptic Operators and Lie Groups*, Oxford University Press, 1991.

[85] W. Rudin, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Springer-Verlag, 1980.

[86] C. Sadosky, *Interpolation of Operators and Singular Integrals: An Introduction to Harmonic Analysis*, Marcel Dekker, 1979.

[87] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, International Mathematics Research Notices No. 2 (1992), 27–38.

[88] S. Semmes, *Differentiable function theory on hypersurfaces in* $\mathbf{R}^n$ *(without bounds on their smoothness)*, Indiana University Mathematics Journal **39** (1990), 985–1004.

[89] S. Semmes, *Bilipschitz mappings and strong* A_∞ *weights*, Annales Academiæ Scientiarum Fennicæ Mathematica Ser. A.I. Math. **18** (1993), 211–248.

[90] S. Semmes, *Good metric spaces without good parameterizations*, Revista Matemática Iberoamericana **12** (1996), 187–275.

[91] S. Semmes, *On the nonexistence of bilipschitz parameterizations and geometric problems about* A_∞ *weights*, Revista Matemática Iberoamericana **12** (1996), 337–410.

[92] S. Semmes, *Finding curves on general spaces through quantitative topology, with applications for Sobolev and Poincaré inequalities*, Selecta Mathematica (N.S.) **2** (1996), 155–295.

[93] S. Semmes, *Some remarks about about metric spaces, spherical mappings, functions and their derivatives*, Publicacions Matemàtiques **40** (1996), 411–430.

[94] S. Semmes, *Analysis on metric spaces*, in *Harmonic Analysis and Partial Differential Equations: Essays in Honor of Alberto Calderón* (M. Christ, C. Kenig, and C. Sadosky, eds.), 285–294, University of Chicago Press, Chicago, 1999.

[95] S. Semmes, *Metric spaces and mappings seen at many scales*, appendix in [52].

[96] S. Semmes, *Some Novel Types of Fractal Geometry*, Oxford University Press, 2001.

[97] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Revista Matemática Iberoamericana **16** (2000), 243–279.

[98] N. Shanmugalingam, *Harmonic functions on metric spaces*, Illinois Journal of Mathematics **45** (2001), 1021–1050.

[99] E. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Annals of Mathematics Studies **63**, Princeton University Press, 1970.

[100] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[101] E. Stein, *Boundary Behavior of Holomorphic Functions of Several Complex Variables*, Princeton University Press, 1972.

[102] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[103] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[104] R. Strichartz, *Analysis on fractals*, Notices of the American Mathematical Society **46** (1999), 1199–1208.

[105] A. Telcs, *Volume and time doubling of graphs and random walks: the strongly recurrent case*, Communications on Pure and Applied Mathematics **54** (2001), 975–1018.

[106] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, 1986.

[107] W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge University Press, 2000.

[108] N. Varopoulos and S. Mustapha, *Analysis on Lie Groups*, Cambridge University Press, 2001.

[109] N. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, 1992.

[110] K. Yosida, *Functional Analysis*, Springer-Verlag, 1965.