Algorithmic classification of 3-manifolds and knots

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1. Introduction

The following is known as *the recognition problem for* 3*-manifolds:*

Does there exist an algorithm to decide whether or not two given 3*-manifolds are homeomorphic?*

Why is this problem important? It is because the positive answer would imply the existence of algorithmic classification of 3-manifolds. Indeed, one can easily construct an algorithm which enumerates step by step all compact 3-manifolds. Using it, we could create a list $M_1, M_2, \dots$ of all 3-manifolds without duplicates by inquiring if each next manifold has been listed before. It is this list that is considered as a *classifying list* of 3-manifolds. Certainly, this is a classification in a very week sense; the knowledge that the classifying list exists does not help to answer questions. It is the proof of the existence that is important since the search for it would inevitably lead one to deeper understanding of the intrinsic structure of 3-manifolds.

Our goal is to overview the positive solution of the above problem for the class of so-called *sufficiently large* 3-manifolds. This case is especially important, since it implies the positive solution of the algorithmic classification problem for knots, one of the most intriguing problems of low-dimensional topology.

Recall that a *knot* is a circle embedded in S^3. Two knots are *equivalent*, if there is an isotopy $S^3 \to S^3$ taking one knot to the other. Knots are usually presented by *knot diagrams*, i.e., by generically immersed plane curves such that at each crossing point it is shown which strand goes over.

If we know at advance that two given knots are equivalent, then one can rigorously prove that by certain local modifications (called *Reidemeister moves*) that transform one diagram to the other. If they are distinct, then sometimes one can prove that by calculating different polynomial or numerical invariants. But what can we do, if both potentially infinite procedures (comparing the knots via Reidemeister moves and via knot invariants) do not stop? The right strategy consists in considering knot complement spaces, which are sufficiently large 3-manifolds.

The history of the positive solution of the recognition problem for sufficiently large 3-manifolds is very interesting. In 1962 W. Haken suggested an approach

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for solving the problem [3]. However, his proof contained a heavy gap. Thanks to efforts of several mathematicians, to the early seventies a crucial obstacle was singled out, and, when in 1978 G. Hemion overcame it [5], it was broadly announced that the problem was solved [8], [14]. Later on many topologists used extensively this result.

Let me explain my interest in the problem. Trying to understand in detail the proof, I discovered that there was no written complete proof at all. All papers and even books ([8], [14], [6]) devoted to this subject were written according to the same scheme: they contained a description of Haken's approach, of the obstacle, of Hemion's result, and the claim that these three ingredients give a proof. But no paper contained a proof of the claim! I undertook an investigation of the question and came to the following conclusions:

1. The statement that the recognition problem for sufficiently large 3-manifolds is algorithmically solvable is true.

2. There is another obstacle of similar nature that cannot be overcome by the same tools as the first one.

3. It can be overcome by using an algorithmic version of W. Thurston's theory of surface homeomorphisms that appeared only in 1995 [1].

Thus for more than 15 years mathematicians relied on an unproven theorem. In this paper we fill the gap by describing a modified proof that is based on the same ideas but is much shorter and simpler than the original one. The paper is based on my talk given at Caen university during my stay at IHES in January 1999.

2. Haken's theory of normal surfaces

The theory of normal surfaces was developed by W. Haken in the early 1960s. Its fundamental importance to the 3-manifold topology cannot be overestimated. Most of the papers on 3-manifolds in recent years are based on or related to it. W. Haken is one of the first topologists who realized that the right strategy to investigate 3-manifolds is to look over surfaces that are contained in it.

Let M be a compact 3-manifold with a fixed triangulation T. Later on we will always assume M is *irreducible* and *boundary irreducible*. It means that every 2-sphere $S \subset M$ must bound a 3-ball, and the boundary of every proper disc $D \subset M$ must bound a disc in ∂M. The restriction has a technical nature. It allows us to avoid considering connected sums, boundary connected sums, and problems related to the Poincaré conjecture.

By a *surface* in M we mean a 2-dimensional proper submanifold of M.

Definition 1. A surface $F \subset M$ is called *normal* if the following holds:

(1) F is in general position with respect to T.
(2) The intersection of F with every tetrahedron consists of discs. Those discs are called *elementary*.
(3) The boundary of every elementary disc crosses at least one edge and crosses each edge at most once.

It is easy to show that for each tetrahedron there are seven types of allowed elementary discs: four triangles and three quadrilaterals, which correspond to plane sections of the tetrahedron (see Fig. 1). Note that elementary discs as

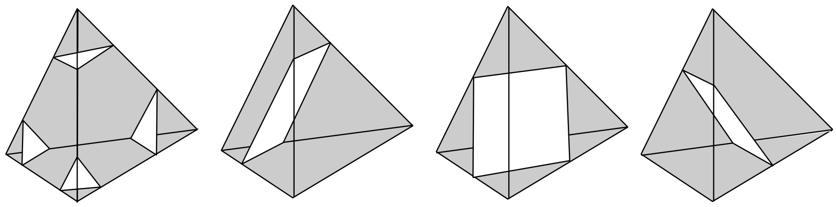

Figure 1. Seven plane sections of the tetrahedron

well as normal surfaces are usually considered up to *normal isotopy* of M, which preserves the triangulation.

It turns out that the set $\mathcal{N}$ of all normal surfaces in M (considered up to normal isotopy) possesses the following two non-formal properties:

(1) $\mathcal{N}$ is informative in the sense that it contains representatives of all interesting classes of surfaces in M. Certainly, the notion of an interesting class depends heavily on the problem we are trying to solve. For our purposes interesting surfaces are *incompressible* ones, see below.
(2) $\mathcal{N}$ admits a more or less explicit description.

As we have mentioned earlier, knowledge of surfaces contained in a given 3-manifold is useful for understanding its structure. Let us think a little on surfaces that are contained in R^3. It is known that any such surface can be obtained from a 2-sphere in R^3 by successive addition of tubes. The tubes may be knotted and linked (see Fig. 2), and run inside each other.

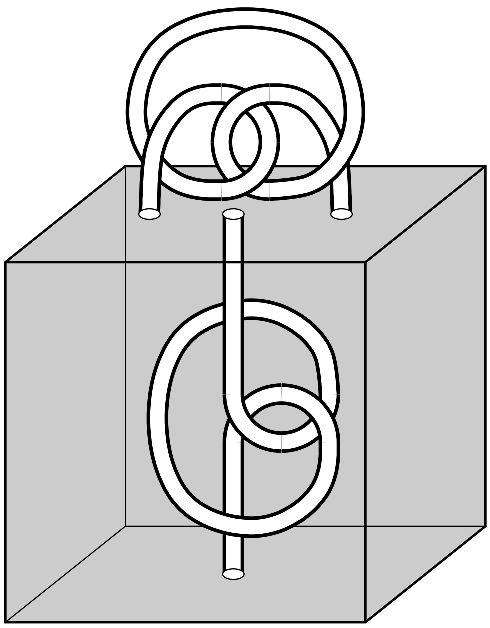

Figure 2. Knotted tubes

Now let us return to the case of arbitrary 3-manifold M. The same procedure (adding tubes) works here as well, but all surfaces obtained in this way seem to be not very interesting. One possible explanation is that all surfaces that are contained in R^3 are contained in every 3-manifold and hence carry no information on M. Thus surfaces without tubes are most interesting. To give a formal description of surfaces without tubes, note that if the tubes are present, then the meridional disc D of the last attached tube meets the surface only along ∂D. We come naturally to the notion of incompressible surface. By definition, a surface F in a 3-manifold M is *incompressible*, if the boundary of every disc $D \subset M$ with $D \cap F = \partial D$ bounds a disc in F.

Similarly, a *boundary incompressible* proper surface F satisfies the following property: for every disc $D \subset M$ which meets F along an arc $l \subset \partial D$ and meets ∂M along the remaining arc of ∂D, the arc l is trivial in F, i.e., cuts off a disc from F.

Proposition 1. *If an incompressible boundary incompressible surface F in an irreducible boundary irreducible triangulated 3-manifold contains no spherical or disc components, then it is isotopic to a normal surface.*

The proof is easy: we simply improve the intersection of F with every tetrahedron Δ^3 of the triangulation by isotopy of F until getting conditions 1 - 3 of Definition 1.

Corollary 1. (First surface finiteness theorem). *Let F be an incompressible closed orientable surface in a triangulated irreducible orientable 3-manifold M. Assume that F contains no spherical and no parallel components. Then $\#(F) \leqslant 10t$, where $\#(F)$ is the number of connected components of F and t is the number of tetrahedra in the triangulation.*

Proof. By Proposition 1 we may assume that F is normal. Connected components of the intersection of F with any tetrahedron Δ^3 of the triangulation are called *patches*. A patch is called *good*, if it lies between two parallel neighboring patches. It is easy to see that each tetrahedron contains no more than 10 bad patches. Since each component of F must contain at least one bad patch, $\#(F) \leqslant 10t$. □

To present an explicit description of $\mathcal{N}$, denote by $E_1, E_2, \ldots, E_n$ all the types of elementary discs in all the tetrahedra. Here $n = 7t$, where t is the number of tetrahedra in T. To each normal surface F we assign an n-tuple $\bar{x}(F) = (x_1, x_2, \ldots, x_n)$ of nonnegative integer numbers in the following natural way: we take the number of elementary discs (triangles or quadrilaterals) of each type E_i in the intersection of F with the tetrahedra. Obviously, any two surfaces having the same vector are normally isotopic. This gives us a parameterization of $\mathcal{N}$ by a subset of the set of all points in R^n with nonnegative integer coordinates. Already this parameterization can be considered as an explicit description of $\mathcal{N}$, but we want more.

Consider an angle of a common triangle face of two tetrahedra Δ^3_1 and Δ^3_2 of T. Each of the tetrahedra contains two types of elementary discs that intersect both sides of the angle: one triangle and one quadrilateral. Let $E_i, E_j \subset \Delta^3_1$ and $E_k, E_l \subset \Delta^3_2$ be the corresponding types. Then we write the equation $x_i + x_j = x_k + x_l$. Doing the same for all the angles, we get the so-called *matching system* of $3m$ linear homogeneous equations, where m is the number of triangles in the interior of M (if M is closed, then $m = 2t$). Clearly, for any normal surface F the n-tuple $\bar{x}(F) = (x_1, x_2, \ldots, x_n)$ is a solution of the matching system. Such solutions are called *admissible.* Thus we have a parameterization of $\mathcal{N}$ by admissible solutions of the matching system. Moreover, algebraic summation of solutions has a geometrical meaning (see [4], [2]), so it makes sense to take sums of surfaces. Note that the Euler characteristic is additive with respect to the summation: if $F = F_1 + F_2$, then $\chi(F) = \chi(F_1) + \chi(F_2)$.

It seems difficult to apply these facts to solving algorithmic problems, since the set of admissible solutions is infinite. However, the set has a finite basis consisting of fundamental solutions. We say that a nonnegative integer solution to the matching system is *fundamental*, if it cannot be presented as a nontrivial sum of two other nonnegative integer solutions.

Proposition 2. *The set of fundamental solutions of any system of linear homogeneous equalities with integer coefficients is finite, and can be constructed algorithmically.*

The proof of this purely algebraic fact is based on the observation that all fundamental solutions are contained in a compact region of R^n, see, for example, [4], [2].

Let F be an incompressible torus or an incompressible boundary incompressible annulus in a 3-manifold M. Then F is called *essential*, if F is not parallel to a surface (torus or annulus) in ∂M. It is convenient to measure *the complexity* of an arbitrary proper surface $F \subset M$ by the number $c(F) = -\chi(F)$, where $\chi(F)$ is the Euler characteristic.

Corollary 2. (Second surface finiteness theorem). *Assume that an irreducible boundary irreducible* 3*-manifold M contains no essential tori and annuli. Then for any number c there exist only finitely many incompressible boundary incompressible surfaces in M of complexity $\leqslant c$ (up to isotopy). The surfaces can be constructed algorithmically.*

Proof. It follows from the main result of [10] that any incompressible boundary incompressible surface F in M can be presented as an integer linear combination $\sum_i k_i F_{i_1}, k_i \geqslant 0$, of fundamental surfaces F_i such that F_i are also incompressible and boundary incompressible. Moreover, since M is irreducible and boundary irreducible, among the surfaces F_i there are no 2-spheres, projective planes, and discs. By the assumption, M contains no essential tori and annuli. Therefore $c_i = c(F_i) > 0$ for all i. The conclusion of the corollary follows from the evident fact that for any number c there are only finitely many integer linear combinations $\sum_i k_i c_i, k_i \geqslant 0$, which do not exceed c. □

3. An idea of classifying sufficiently large 3-manifolds.

A 3-manifold M is called *sufficiently large*, if it contains a two-sided closed incompressible surface $F \neq S^2, RP^2$. The class of sufficiently large 3-manifolds is "sufficiently large". For example, every irreducible 3-manifold M with $\partial M \neq \varnothing$ is either sufficiently large or homemomorphic to a solid pretzel. Every irreducible closed 3-manifold M having infinite first homology group $H_1(M; Z)$ is also sufficiently large. Indeed, to construct an incompressible surface in M, consider a map $f\colon M \to S^1$ that induces a surjective homomorphism $H_1(M; Z) \to Z$. The inverse image of a regular point $\{*\} \in S^1$ contains a nonseparating surface F in M. After compressing F, i.e., after cutting all tubes, we get a nonseparating incompressible surface.

A similar construction shows that if $M \neq B^3$ is irreducible and $\partial M \neq \varnothing$ is incompressible, then M contains an incompressible boundary incompressible surface which does not separate M.

Definition 2. An orientable 3-manifold is called *Haken*, if it is irreducible, boundary irreducible, and sufficiently large.

Main Theorem. *There exists an algorithm to decide whether or not two Haken 3-manifolds are homeomorphic.*

We present here a modified version of Haken's original idea of the proof. Let M be a Haken 3-manifold. If M is closed, we choose a minimal (i.e., having the maximal Euler characteristic) incompressible surface $F \neq S^2, RP^2$ in M, which we will consider as *a partition wall.* Here is the only place where we use the assumption that M is sufficiently large. If $\partial M \neq \varnothing$, we take $F_0 = \partial M$.

Then we begin to erect new and new partition walls inside M. Each next wall F_{k+1} is a minimal incompressible boundary incompressible surface attached to the union $P_k = \cup_{0 \leqslant i \leqslant k} F_i$ of the previous walls along its boundary curves. The boundary curves of F_{k+1} should be in general position with respect to the set of singular points of P_k. This requirement guarantees us that each P_k is *a simple polyhedron* in the sense that it has the simplest possible singularities: *triple lines* (where one wall is attached to another one) and *true vertices* (crossing points of triple lines). The polyhedron P_k decomposes M into parts called *chambers.*

At last we get a 2-dimensional simple polyhedron P such that (1) all chambers of the pair (M, P) are 3-balls, and (2) the set of nonsingular points of P consists of open 2-cells. Any simple subpolyhedron of M possessing properties 1, 2 above is called a *special skeleton* of M. P can be considered as a kind of *hierarchy* for M (see [3]).

Denote by $\mathcal{P}(M)$ the set of all special skeletons of M considered up to homeomorphisms of M. It follows from the definition that the set $\mathcal{P}(M)$ is *characteristic* in the following sense:

M and N are homeomorphic $\iff$ $\mathcal{P}(M)$ and $\mathcal{P}(N)$ do coincide, i.e., there is a bijection $\mathcal{P}(M) \to \mathcal{P}(N)$ taking each polyhedron to a homeomorphic one.

Haken's idea consists in an attempt to replace the infinite set $\mathcal{P}(M)$ by a finite algorithmically constructible subset $\mathcal{P}_0(M)$ having the same characteristic property. This would reduce the recognition problem for Haken 3-manifolds to algorithmic comparing two finite sets of 2-dimensional polyhedra, which is trivial.

Below we will describe a sort of transformations of simple subpolyhedra called *extension moves*. Let P_k be a simple subpolyhedron of a Haken 3-manifold$M$. Each extension move transforms P_k to a bigger simple subpolyhedron P_{k+1} of M. The following conditions should be satisfied:

C_1. The number of different extensions of P_k is finite (up to homeomorphisms of the pair (M, P_k)), and all of them can be constructed algorithmically;

C_2. Any sequence $P_0, P_1, P_2 \dots$, where $P_0 = \partial M$ and each P_{k+1} is an extension of P_k, is finite;

C_3. If P_k has no extensions, then P_k is a special skeleton of M.

Proof of the Main Theorem. (Under assumption that the extension moves had been described). Suppose that M is a Haken 3-manifold. Let us apply to P_0 step by step all extension moves as long as possible. It follows from conditions $C_1 - C_3$ that we get a finite set $\mathcal{P}_0(M)$ of special skeletons of M depending only on the topological type of M. On the other hand, if for two Haken manifolds M, N the sets $\mathcal{P}_0(M)$ and $\mathcal{P}_0(N)$ contain at least one pair of homeomorphic polyhedra, then M and N are homeomorphic. Indeed, we can extend a homeomorphism of the special skeletons to a homeomorphism of the manifolds by applying the cone construction to each ball chamber. It follows that $\mathcal{P}_0(M)$ is characteristic. □

It remains to construct extension moves satisfying conditions $C_1 - C_3$. The idea consists in adding new surfaces (partition walls) having minimal complexity. Corollary 2 tells us that the first difficulty in realizing the idea is the presence of essential tori and annuli (they hinder us from having only a finite number of new walls). The famous JSJ-decomposition theorem helps us to overcome it.

4. Jaco-Shalen-Johannson decomposition.

It happens often in mathematics that a proof of some statement becomes easier if we formulate the statement in more general form. Having this in mind, we will consider 3-*manifolds with boundary pattern*, where a boundary pattern is a fixed graph (one-dimensional polyhedron) in the boundary of the manifold [7]. In particular, every manifold can be considered as a manifold with the empty boundary pattern. A map between manifolds with boundary pattern is a map $M_1 \to M_2$ which takes $\Gamma_1 \to \Gamma_2$, i.e., a map of pairs $(M_1, \Gamma_1) \to (M_2, \Gamma_2)$.

Manifolds with boundary pattern appear naturally by realizing Haken's idea: each chamber $Q \subset M, Q \cap P_k = \partial Q$, is a manifold with boundary pattern consisting of triple lines and vertices of P_k which lie in ∂Q.

Let (M, Γ) be a manifold with boundary pattern. A surface $F \subset M$ is called *proper* if $F \cap \partial M = \partial F$ and ∂F is in general position with respect to Γ. The complexity $c(F)$ of F is given by the formula $c(F) = -\chi(F) + \#(F \cap \Gamma)$, where $\#(F \cap \Gamma)$ is the number of points in $F \cap \Gamma$. A surface F is *clean* if $F \cap \Gamma = \varnothing$.

The notions of irreducible boundary irreducible 3-manifold, of incompressible boundary incompressible surface, and of essential torus or annulus admit natural generalizations to the case of manifold M with boundary pattern Γ. In particular, any essential torus must be not parallel to a clean torus in ∂M, and any clean essential annulus must be not parallel to an annulus A in ∂M such that $A\cap\Gamma$ is a collection of several parallel copies of the core circle of A. Haken's theory of normal surfaces works here as well. Corollary 2 also remains true: if an irreducible boundary irreducible 3-manifold M with boundary pattern Γ contains no essential tori and clean essential annuli, then for any number c there exist only finitely many incompressible boundary incompressible surfaces in M of complexity $\leqslant c$.

Definition 3. Let M be an irreducible boundary irreducible 3-manifold with boundary pattern Γ, and F an essential torus or clean essential annulus in M. Then F is *rough* if any other essential torus and any other clean essential annulus in M is cleanly isotopic to a surface disjoint from F.

The terminology is related to an observation that everybody would avoid contacts with a rough person.

Let M be an irreducible boundary irreducible 3-manifold M with boundary pattern Γ. A finite collection of disjoint pairwise nonparallel rough tori and annuli in M is a *JSJ-system* if it is maximal, i.e., if any other rough torus or annulus in M is isotopic to a member of the collection.

We formulate now a version of the usual JSJ-decomposition theorem for manifolds with boundary pattern (JSJ stands for W. Jaco, P. Shalen, and K. Johannson [11] ,[7]). In the case of empty boundary pattern an easy proof can be found in [13].

JSJ-decomposition Theorem. *For any irreducible boundary irreducible 3-manifold M with boundary pattern Γ the JSJ-system exists and is unique up to clean isotopy.* □

Let S be a JSJ-system in (M,Γ). Then S decomposes M into a collection $\{Q_i\}$ of compact manifolds called 3-*components* of the pair (M,S). We equip every Q_i with a boundary pattern $\Delta_i \subset \partial Q_i$ consisting of the portion $\partial Q_i \cap \Gamma$ of the pattern for M, and of traces in ∂Q_i of the boundary circles of annuli from S.

Let us investigate the structure of Q_i. Note that the JSJ-system for (Q_i,Δ_i) is empty. It means that all rough tori and annuli in Q_i are inessential. It may happen that (Q_i,Δ_i) contains no essential tori and annuli at all. Then (Q_i,Δ_i) is called *simple*. On the other hand, Q_i may contain essential tori and annuli, which necessarily are not rough. Let us present two basic examples.

Example 1. Let Q be a Seifert manifold fibered over a surface F. Suppose that Q is equipped with boundary pattern Δ consisting of a collection of fibers. Then, with a few exceptions, Q contains tori or annuli which are essential but not rough. The exceptions are fibered solid tori and manifolds fibered over S^2 with $\leqslant 3$ exceptional fibers.

Example 2. Let Q be an I-bundle, i.e., an orientable direct or skew product of a surface F and an interval. Suppose that Q is equipped with a boundary pattern Δ consisting of a nonempty collection of nontrivial circles in each annulus of the induced I-bundle over ∂F. Then, with a few exceptions, Q contains annuli that are essential but not rough. The exceptions are I-bundles over surfaces with nonnegative Euler characteristics.

Let us introduce two types of essential annuli in manifolds with boundary pattern.

Definition 4. An essential annulus A in a 3-manifold Q with boundary pattern Δ is called *longitudinal* if any clean incompressible boundary incompressible annulus $A_1 \subset Q$ is cleanly isotopic to an annulus A'_1 such that $\partial A \cap \partial A'_1 = \varnothing$. Otherwise A is called *transverse.*

Note that all essential rough annuli are longitudinal, while transverse annuli are never rough.

Proposition 4. *Suppose that a manifold Q with boundary pattern Δ contains no rough annuli and tori. If Q contains an essential torus or a longitudinal annulus, then Q is a Seifert manifold (as in Example 1). If Q contains a transverse annulus, then Q is an I-bundle (as in Example 2).*

Proof. To be definite, assume that Q contains an essential annulus A. Since A is not rough, there is an essential annulus A' such that $A \cap A'$ is a nonempty collection of nontrivial circles or radial segments. Then $A \cup A'$ fibers into circles in the first case and into segments in the second, and so does a regular neighborhood N of $A \cup A'$. Applying the same procedure to an essential annulus or torus contained in ∂N, we extend the fibration to a larger submanifold of Q. The "no rough annuli" assumption tells us that there are no obstructions for extending the fibration over the whole Q. □

Corollary 3. *Any irreducible boundary irreducible manifold M with boundary pattern Γ contains only finitely many incompressible tori and longitudinal annuli up to homeomorphisms of M fixed on ∂M.*

Proof. Since the behavior of essential tori and annuli in Seifert manifolds is well known (up to homeomorphisms, there are only finitely many of them), the conclusion follows from the fact that every such surface can be isotoped into a Seifert part of M. □

5. Realization of Haken's idea.

Let M be a Haken 3-manifold. We describe an algorithmic construction of a finite characteristic set $\mathcal{P}_0(M)$ of special skeletons. Starting with $P_0 = \partial M$, we will erect new partition walls. Let P_k be a simple subpolyhedron of M, obtained in an intermediate step of the construction. Consider a chamber (Q_i, Γ_i) of (M, P_k).

Extension move E_1. *Assume that (Q_i, Γ_i) contains an essential torus T or a longitudinal annulus A. Then we enlarge P_k by adding T or A, respectively.*

Although (Q_i, Γ_i) is boundary irreducible as a manifold with boundary pattern, ∂Q_i can be compressible if we forget about Γ_i.

Extension move E_2. *Suppose that (Q_i, Γ_i) contains no essential tori and clean annuli, ∂Q_i is compressible, and Q_i is not a solid torus with a clean longitude on the boundary. Then we extend P_k by adding a nontrivial compressing disc having the smallest complexity.*

Extension move E_3. *Assume that Q_i is not a 3-ball, (Q_i, Γ_i) contains no essential tori and clean annuli, and that ∂Q_i is incompressible. Then Q_i, considered as a manifold without boundary pattern, contains an incompressible boundary incompressible surface $F \neq S^2, RP^2$ that, if $\partial M \neq \varnothing$, does not* *split M. Among all such surfaces we choose a surface of the minimal complexity and insert it as a new partition wall.*

Let us now perform moves $E_1 - E_3$ as long as possible. Since each time we get simpler chambers, the process terminates after a finite number of steps. By Corollary 2 for manifolds with boundary pattern and Corollary 3, at each step we have only finitely many choices of new partition walls, and the walls can be constructed algorithmically. Therefore, using all of them (and multiplying (M, P_k) in corresponding number of exemplars), we get a finite set of simple subpolyhedra of M.

Let us analyze the structure of chambers for a polyhedron P_k obtained at this stage. By construction, we can have chambers of the following three types: *balls, I-bundles, and solid tori having a clean longitude in the boundary.*

Ball type chambers are exactly what we want. Our goal is to decompose into balls the chambers of the other two types.

Denote by U the union of the I-bundle chambers. Then for any connected component U_0 of U one of the following holds:

(a) U_0 is an I-bundle $F\tilde{\times}I$, such that $F\tilde{\times}\partial I \subset \partial M$;
(b) U_0 is a Stallings manifold M_f obtained from $F \times I$, where F is a surface, by identifying $F \times \{0\}$ and $F \times \{1\}$ via a homeomorphism $f\colon F \times \{0\} \to F \times \{1\}$;
(c) U_0 is a quasi-Stallings manifold $M_{\alpha,\beta}$ obtained from $F \times I$ by identifying $F \times \{0\}$ with $F \times \{0\}$ and $F \times \{1\}$ with $F \times \{1\}$ via free involutions $\alpha\colon F \times \{0\} \to F \times \{0\}$ and $\beta\colon F \times \{1\} \to F \times \{1\}$, respectively.

Extension move E_4. *Assume that a connected component U_0 of U is an I-bundle $F\tilde{\times}I$. Then we decompose it into balls by inserting discs of the type $l\tilde{\times}I$, where l is a proper arc in F.*

This can be done in a finite number of ways. Indeed, up to homeomorphisms $U_0 \to U_0$ fixed on $\partial F\tilde{\times}I$, there are only finitely many such discs. On the other hand, since $F\tilde{\times}\partial I \subset \partial M$, any such homeomorphism $U_0 \to U_0$ can be extended to a homeomorphism $M \to M$.

Let U_0 be a Stallings manifold as in (b). The set of homeomorphisms $U_0 \to U_0$ can be very poor, so the above trick does not work. After a little thinking we come to the conclusion:

In this situation Haken's idea does not work, and one should find an independent solution of the recognition problem for Stallings manifolds.

It is the key problem that was overlooked by W. Haken and solved by G. Hemion. Note that two Stallings manifolds M_f, M_g with the same fiber F are fiber-preserving homeomorphic if and only if the monodromy maps $f, g\colon F \to F$ are conjugate (modulo isotopy). Since in our situation the manifolds contain no essential tori and annuli, one can assume that f and g are pseudo-Anosov, i.e., admit no essential periodic curves.

Theorem. (Hemion [5]). *For any two pseudo-Anosov homeomorphisms $f, g\colon F \to F$ there exists an algorithmically constructible finite set $\mathcal{H}(f, g) = \{h_1, \ldots, h_m\}$ of homeomorphisms $F \to F$ such that the following holds:*

(1) f and g are conjugate $\Longleftrightarrow g = h_i f h_i^{-1}$ for some $h_i \in \mathcal{H}(f, g)$;

(2) Every homeomorphism $h\colon F \to F$ *that conjugates* f *to* g *has a form* $h = h_j f^n$*, where* $h_j \in \mathcal{H}(f,g)$ *and* n *is an integer.* □

This theorem solves the recognition problem for Stallings manifolds by reducing it to simple checking whether some h_i conjugates f to g. Simultaneously it helps us to subdivide any Stallings chamber into balls in essentially canonic way (see [12] for details).

Next EXTENSION MOVE E_5 consists in doing that.

Now let U_0 be a quasi-Stallings manifold as in (c). Just as above, we must find an independent solution of the recognition problem for such manifolds. Hemion's theorem is insufficient for the purpose (that fact was overlooked in [8], [14], [6]). Note that two quasi-Stallings manifolds $M_{\alpha,\beta}, M_{\alpha',\beta'}$ with the same fiber F are fiber-preserving homeomorphic if and only if there exists a homeomorphism $\psi\colon F \to F$ such that $\psi\alpha\psi^{-1} = \alpha'$ and $\psi\beta\psi^{-1} = \beta'$. Multiplying the equalities, we get the equality $\psi\alpha\beta\psi^{-1} = \alpha'\beta'$, which has a clear geometric meaning: it determines a homeomorphism $M_{\alpha\beta} \to M_{\alpha'\beta'}$ of Stallings manifolds $M_{\alpha\beta}, M_{\alpha'\beta'}$ that double cover $M_{\alpha,\beta}, M_{\alpha',\beta'}$, respectively. By Hemion's theorem there are only finitely many candidates for ψ (modulo postcomposing with $(\alpha\beta)^n$), and they can be enumerated algorithmically. Thus we can assume that $\psi = \psi_0(\alpha\beta)^n$, where ψ_0 is known.

Let us transform the first equality:

$$\psi_0(\alpha\beta)^n\alpha(\alpha\beta)^{-n}\psi_0^{-1} = \alpha' \iff \psi_0(\alpha\beta)^n\alpha(\beta\alpha)^n\psi_0^{-1} = \alpha' \iff$$
$$\psi_0(\alpha\beta)^{2n}\alpha\psi_0^{-1} = \alpha' \iff (\alpha\beta)^{2n} = \psi_0^{-1}\alpha'\psi_0\alpha \quad .$$

We have run into the following problem:

Does there exist an algorithm to decide whether one of two given pseudo-Anosov homeomorphisms $f, g\colon F \to F$ *is isotopic to an integer power of another? (Here* f *and* g *stand for* $\alpha\beta$ *and* $\psi_0^{-1}\alpha'\psi_0\alpha$*, respectively; both of them are known.*

The positive answer to this question would imply a solution of the recognition problem for quasi-Stallings manifolds. It turns out that the answer can be obtained by Thurston's theory of surface homeomorphisms [**?**], more precisely, by its algorithmic version [1]. Let us comment this.

Every pseudo-Anosov homeomorphism $f\colon F \to F$ has the so-called stretching factor $\lambda(f)$ possessing the following properties:

(1) $\lambda(f)$ is an algebraic number greater than 1, and $\lambda(f^n) = \lambda(f)^n$;
(2) Isotopic homeomorphisms have the same stretching factor;
(3) $\lambda(f)$ can be calculated algorithmically.

These properties allow us to find n such that $\lambda(f^n) = \lambda(g)$ (if it exists), and reduce the problem to a routine checking whether f^n and g are isotopic. Certainly, the same should be done for the pair (g, f).

Having solved the recognition problem for quasi-Stalling manifolds, we can introduce next EXTENSION MOVE E_6, which consists in subdividing into balls the quasi-Stallings chambers.

Applying moves $E_1 - E_6$ as long as possible, we get a partition of M into balls and solid tori having clean longitudes. Let U_0 be a connected component of the union of the solid tori. Then U_0 is a Seifert manifold without exceptional fibers, i.e., a direct or twisted product of a surface and a circle. It is easy to subdivide U_0 into balls by inserting a cross-section of the fibration. To be sure that it can be done in a finite number of ways, we take a cross-section S whose boundary curves intersect singular points of P_k in a minimal number of points. The intersection of S with each solid torus is a meridional disc of the torus.

In general, the polyhedron $P_k \cup S$ is not simple, since it may contain four-fold lines. To transform it to a special skeleton we replace S by a collection of disjoint meridional discs of the tori such that the discs are close and parallel to S. The next and last EXTENSION MOVE E_7 consists in decomposing U_0 into balls by choosing a minimal cross-section and perturbing it as above.

By construction, moves $E_1 - E_7$ possess properties $C_1 - C_3$, see Section 3. This completes the proof of Main Theorem. □

6. Concluding remarks.

Remark 1. Let us show how Main Theorem gives us the algorithmic classification of knots. By the same philosophy as in the introduction, it is sufficient to construct a recognition algorithms for knots.

Let K be a knot. Denote by $C(K)$ the complement manifold obtained from S^3 by removing an open tubular neighborhood of K. We supply $C(K)$ with a boundary pattern Γ consisting of a meridional circle of K. It is easy to see that two nontrivial knots K_1, K_2 are equivalent if and only if $C(K_1), C(K_2)$, considered as manifolds with boundary pattern, are homeomorphic. It remains to note that both manifolds are Haken, and apply Main Theorem. Certainly, the same method classifies links in S^3.

Remark 2. The technic of hierarchies described above is sufficient to solve nearly all problems about Haken manifolds. For example, one can calculate the Heegaard genus [9], solve the word problem in the fundamental group [15], and so on.

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