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LEONARD DICKSON'S *HISTORY OF THE THEORY OF NUMBERS*: AN HISTORICAL STUDY WITH MATHEMATICAL IMPLICATIONS

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ABSTRACT. — In 1911, the research mathematician Leonard Dickson embarked on a *historical* study of the theory of numbers which culminated in the publication of his three-volume *History of the Theory of Numbers*. This paper discusses the genesis of this work, the historiographic style revealed therein, and the mathematical contributions which arose out of it.

RÉSUMÉ. — *HISTORY OF THE THEORY OF NUMBERS* DE LEONARD DICKSON: ÉTUDE HISTORIQUE AVEC DES PROLONGEMENTS MATHÉMATIQUES. — En 1911, le mathématicien Leonard Dickson s'est lancé dans une étude *historique* de la théorie des nombres, qui a culminé avec la publication de son *History of the Theory of Numbers* en trois volumes. Notre étude examine la genèse de ce travail, l'approche historiographique qui la sous-tend et les contributions mathématiques qui en découlent.

In 1911, only a decade into what would become a forty-year-long career in the mathematics department at the University of Chicago, Leonard Dickson had a résumé which solidly identified him as a distinguished mathematician. He had, for example, authored roughly 150 mathematical papers (primarily in group theory at this point) and three books, served as editor of the *American Mathematical Monthly* from 1902 to 1908 and recently assumed this post for the *Transactions of the American Mathematical Society*, and passed swiftly through the ranks from assistant to associate to full professor at one of the premiere mathematics institutions in this country. Yet, in 1911, he threw what seems a rather twisted turn into his professional plans by pursuing a *historical* project which would

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interrupt his pure mathematical research for the better part of nine years. That investigation? The title of this paper gives it away: he embarked on a study of the history of the theory of numbers. This almost ninety-year-old decision raises (raised?) a flurry of questions. In this paper, we focus on the genesis of this work, the historiographic style revealed therein by Dickson, and one of the surprising mathematical contributions which arose out of it. An understanding of the origins of this work, however, begins with an understanding of its author, the then thirty-seven-year-old Leonard Dickson.

Born in Independence Iowa in 1874, Dickson spent his boyhood in Cleburne, Texas and ultimately attended the University of Texas for his undergraduate and master's education.[1] With his master's degree in hand and two years of teaching experience under his belt, Dickson chose the strong Eliakim Hastings Moore, Oskar Bolza, and Heinrich Maschke triumvirate at the young University of Chicago over the up-and-coming Harvard with William Fogg Osgood and Maxime Bôcher as the place to pursue his doctorate. Dickson's mathematical career would ultimately hinge on this decision [Fenster 1997, pp. 9–13].

At the time, Chicago, with its sights set on emulating the German tradition of scholarship, stood in marked contrast to most American institutions. Specifically, Moore, Bolza and Maschke formed the core of the original far-sighted Chicago Mathematics Department which promoted *both* research and teaching and which emphasized in its graduate program the training of future productive researchers [Parshall & Rowe 1994, pp. 363–426], [Fenster 1997, pp. 10–11].

While Dickson pursued a Ph.D. at the young Chicago from 1894 to 1896, the then group-theoretically minded Moore inspired him to write a thesis on (what we would call) permutation groups [Dickson 1897]. Although group theory would remain among Dickson's research interests throughout his career, he would add finite field theory, invariant theory, the theory of algebras and number theory to his repertoire of research interests. Dickson reflected Chicago's influence—particularly that of Moore—in more ways than in his research interests, however. The

[1] [Bell 1938] and [Albert 1955] serve as the standard sources for biographical information on Dickson. This author also consulted [University of Texas 1899, 1914] and [Parshall 1991].

department's sustained commitment to research, high standards for publication, and their vision for the American (as opposed to New England) mathematical community came to permeate Dickson's mathematical persona in these formative years. In the spring of 1900, the Chicago Mathematics Department invited Dickson to join them as an assistant professor. From this position, Dickson made significant contributions to the consolidation and growth of the algebraic tradition in America [Fenster 1997, p. 21]. Specifically, Dickson spent forty years (all but the first two) of his professional career on the faculty at Chicago where he directed 67 Ph.D. students, wrote more than 300 publications, served as editor of the *Monthly* and the *Transactions*, and guided the American Mathematical Society as its President from 1916–1918.

And, yet, this mathematical workhorse, who played billiards and bridge by day and did mathematics from 8: 30 p.m. to 1: 30 a.m. every night [Albers & Alexanderson 1991, p. 377], spent nearly a decade of his career writing a three-volume, 1500-page history of the theory of numbers. The lurking question is: why? Why did Dickson interrupt his own pure investigations of mathematics to write a history of the theory of numbers?

Dickson's most celebrated student, A. Adrian Albert, has suggested that Dickson wrote the book to become more acquainted with number theory. More precisely, Albert wrote, "*Dickson always said that mathematics is the queen of sciences, and that the theory of numbers is the queen of mathematics. He also stated that he had always wished to work in the theory of numbers and that he wrote his monumental* History of the Theory of Numbers *so that he could know all of the work which had been done in the subject*" [Albert 1955, p. 333].

Dickson's developing research interests substantiate this claim. Of the 200 papers he wrote prior to 1923, the year he published the third (and final) volume of his *History of the Theory of Numbers* (hereinafter *History*), only ten considered number-theoretic topics.[2] In 1927, however, his pure mathematical researches began to focus on additive number theory, on the ideal Waring theorem, in particular. In a long series of papers, he and his students provided an almost complete verification of the theorem which loosely states that every positive integer is a sum of I integral n-th powers for sufficiently large I. Moreover, Dickson guided

[2] [Albert 1955, pp. 334–345] contains a bibliography of Dickson's work.

twenty-nine of his last thirty-two doctoral students in number-theoretic dissertations [University of Chicago 1931, 1938, 1941]. These twenty-nine students, along with Dickson's contributions to the ideal Waring theorem and three number theory texts he published in 1929, 1930, and 1939 [Dickson 1929, 1930, 1939] seem to indicate that if he intended for his historical study to acquaint him with the subject so that he could work in the field himself, he had certainly accomplished what he set out to do.

In some sense, given the time period under discussion, this connection between the history of mathematics and pure mathematical results comes as no surprise. The early decades of this century, in fact, represented a "*golden age for the history of mathematics*" since "*the historians of mathematics were professional mathematicians working in good mathematics departments*" [Gray 1998, p. 54].[3] Still, however, some members of the mathematical community viewed those who wrote *about* mathematics, in contrast to those who "*d*[*id*] *something*" in mathematics (*i.e.* "*proved new theorems*" or "*added to mathematics*"), as "*second-rate minds*" [Hardy 1940, p. 61]. With a solid reputation as a "*powerful*" [Mac Lane 1992] and prolific research mathematician [Fenster & Parshall 1994, pp. 185–186], Dickson apparently had no qualms about devoting time to the history of mathematics for more than a third of his career.

He may, however, have had other reasons for undertaking this historical work. In his initial letter to the Carnegie Institution seeking interest in the project, for example, Dickson outlined that "[*i*]*t would seem desirable to have undertaken in this country something of the kind done by the British Association, the Deutsche Mathematiker-Vereinigung, etc., in the preparation by specialists of note of extensive Reports each covering an important branch of science... I have already given a solid year's work to such an expository Report on the theory of numbers (integral and algebraic),...*" [Dickson 1911]. Thus the British and German mathematical Report[s], and, in particular, the lack of similar offerings in America,

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[3] [Merzbach 1989, p. 642] also documents that "*with one notable exception* [David Eugene Smith]*,*" the historians of mathematics in America before World War I came from "*those trained in mathematics and allied fields rather than from historians.*" From post-World War I to 1930, the American historians of mathematics had strong ties with—and leadership roles within—the American Mathematical Society [*Ibid.*, p. 650].

may have encouraged Dickson to write his own compendium on the subject of number theory. In the case of graduate training, it was not at all unusual for the American mathematicians to look to the Europeans for ideas [Parshall & Rowe 1994]. The initiative Dickson outlined in his letter to Woodward, however, required not only an acquaintance with the European literature but also an awareness of a perceived void in American publications. Moreover, the opening sentence of his letter seems to suggest that Dickson wanted to raise American mathematics to the European standard in this particular realm.[4]

But Dickson himself gave another—more altruistic—reason for writing what grew into this three volume *History*. In the preface to the second volume, Dickson asserts that he embarked on this historical study because "*it fitted with my conviction that every person should aim to perform at some time in his life some serious useful work for which it is highly improbable that there will be any reward whatever other than his satisfaction therefrom*" [Dickson 1920, p. xxi]. Robert Carmichael extinguished any doubts of Dickson's sincerity in his review of this second volume for the *Monthly*. Carmichael, a number theorist who not only wrote the review of Dickson's *History* for the *Monthly*, but also read the proof sheets for the entire second and third volumes, described Dickson's motivation in similar terms. As Carmichael expressed it, "[*i*]*t is refreshing and inspiring to find a man, when he pauses at a breathing place in the excellent performance of a great task, willing to set forth in a quiet way the fact that he has been moved by the highest and most unselfish ideal of duty*" [Carmichael 1921, p. 78]. In the end, though, as we will see, whether motivated by a desire to acquaint himself with number theory, to publish an American report on the theory of numbers, or to fulfill this "*highest and unselfish ideal of duty,*" this historical initiative led Dickson to one of his most celebrated mathematical contributions.

First, however, let's take a closer look at Dickson's *History* itself. Dickson's view of the role of the historian dictated how he both prepared and wrote his book. As he saw it, "[*w*]*hat is generally wanted* [in a historical study] *is a full and correct statement of the facts, not an*

[4] Throughout his career, Dickson remained avidly committed to establishing standards of excellence for and in the community of American mathematicians. See [Fenster, forthcoming].

historian's personal explanation of those facts. The more completely the historian remains in the background, the better the history. Before writing such a history, he must have made a more thorough search for all the facts than is necessary for the conventional history" [Dickson 1920, p. xx]. For Dickson, this "*thorough*" search required a trip overseas to visit European libraries and collect various number theoretic references. The University of Chicago, apparently, supported this type of international research travel since they granted Dickson a leave of absence. For the necessary funds, Dickson sought travel support for his research from the Carnegie Institution of Washington, one of the new national agencies created to promote what we now call basic research [Reingold & Reingold 1981, p. 7].[5] From a purely pragmatic perspective, Dickson's *History* confirms the importance of recent "*technical innovations*" in the internationalization of science [Parshall 1996, p. 293], [Lehto 1998, pp. 1–2]. Dickson could not have undertaken—much less completed—his *History* without the recent advances of the railroad to take him to the East Coast of the United States, the steamship to carry him across the Atlantic and the telegraph to aid him with his correspondence.

As for the presentation, a typical page from Dickson's *History* reveals the stylistic manifestation of Dickson's historiographic view.

> "*Hrotsvitha, a nun in Saxony, in the second half of the tenth century, mentioned the perfect numbers* 6*,* 28*,* 496*,* 8128*.*
>
> *Abraham Ibn Ezra (1167), in his commentary to the Pentateuch, Ex. 3, 15, stated that there is only one perfect number between any two successive powers of* 10*.*
>
> *Rabbi Josef b. Jehuda Ankin, at the end of the twelfth century, recommended the study of perfect numbers in the program of education laid out in his book '*Healing of Souls*'.*
>
> *Jordanus Nemorarius (1236) stated (in Book VII, props. 55, 56) that every multiple of a perfect or abundant number is abundant, and every divisor of a perfect number is deficient. He attempted to prove (VII, 57) the erroneous statement that all abundant numbers are even.*
>
> *Leonardo Pisano, or Fibonacci, cited in his Liber Abbaci of 1202, revised about 1228, the perfect numbers*
>
> $$2^2(2^2-1)=6,\quad \frac{1}{2}2^3(2^3-1)=28,\quad \frac{1}{2}2^5(2^5-1)=496,$$
>
> *excluding the exponent* 4 *since* 2^4-1 *is not prime. He stated that by proceeding so, you can find an infinitude of perfect numbers*" [Dickson 1919, p. 5].

[5] "*Extending the Frontiers of Science*" serves as the current motto of the Carnegie Institution.

True to his word, Dickson presented what he called "*the facts*". Hrotsvitha mentioned, Ezra stated, Rabbi Josef recommended, Nemorarius stated, Fibonacci cited, *etc.* Dickson adhered to this style throughout the entire three-volume series.

In some cases, however, the *sum total* of the facts departed from the strictly internalistic style followed by Dickson, to use a modern historiographic adjective, and revealed a much broader view of the theory of numbers. As we saw above, Dickson included a twelfth-century rabbi as a contributor to the development of perfect numbers and described his contribution as one who "*recommended the study of perfect numbers in the program of education laid out in his book* 'Healing of Souls.'" The preceeding page included more "*facts*" on the ethical importance of perfect numbers.

> "*Iamblichus (about 283–330)... remarked that the Pythagoreans called the perfect number* 6 *marriage, and also health and beauty (on account of the integrity of its part and the agreement existing in it).*
>
> *Aurelius Augustinus (354–430) remarked that,* 6 *being the first perfect number, God effected the creation in* 6 *days rather than at once, since the perfection of the work is signified by the number* 6...
>
> *Alcuin (735–804), of York and Tour, explained the occurrence of the number* 6 *in the creation of the universe on the ground that* 6 *is a perfect number. The second origin of the human race arose from the deficient number* 8*; indeed, in Noah's ark there were* 8 *souls from which sprung the entire human race, showing that the second origin was more imperfect than the first, which was made according to the number* 6" [Dickson 1919, p. 4].

Hence, as Derrick Lehmer pointed out in his review of this volume for the *Bulletin of the American Mathematical Society*, "*one is struck in glancing through the book by the remarkable combination of superstition, fancy, scientific curiosity, and patient, plodding experiment that has figured in advancing the science of the theory of numbers*" [Lehmer 1920, p. 125]. Dickson may or may not have minded this sort of comment made about his book, but he certainly would have never drawn the conclusion in the book itself. From our perspective today, we may further agree with Lehmer that this book may not be so much a history itself, but, rather, a list of references from which a history might be written [Lehmer 1920, p. 132].

But Dickson—knowingly or otherwise—made other contributions to the history of mathematics. In particular, the history he tucked into his works on the arithmetic of algebras had other aims than the comprehensive fact-reporting of his *History*. In this case, he seemingly intended to use his history as a rhetorical device to persuade the audience of the importance of his own contribution to the theory [Fenster 1998]. Dickson's "*history*" thus introduces issues in the historiography of the history of mathematics. Ivor Grattan-Guinness's "*royal road to me*" view of mathematics seems to characterize Dickson's presentation of the emerging theory of the arithmetics of algebras. As the rather pejorative name implies, these types of accounts focus more on how older theories led to an individual version of a theory than on how the theory developed in its own right [Grattan-Guinness 1990], [Fenster 1998, p. 121].

A consideration of Dickson's historical contributions necessarily calls attention to the interplay between the history of mathematics and mathematical research. In particular, the differences in methodology between the practicing historian and the practicing mathematician come to the fore. If Dickson had written his *History* with the intentions of a historian, for example, he would have included more of a contextual setting for his study and provided historical arguments. He, however, wrote his *History* as a comprehensive literature review, precisely what a mathematician needs. Not surprisingly, Dickson interrupted—but did not stop—his pure mathematical researches to write a compendium of the history of the theory of numbers [Fenster 1999].

In fact, although Dickson had fairly focused research interests for years at a time, he was never one to have his hand in one piece of the mathematical pie at a time. He concurrently pursued his various interests and, in 1920, he brought together two of these seemingly disparate areas of mathematics in one of the most prestigious talks of his career, his plenary address at the International Congress of Mathematics in Strasbourg [Dickson 1921d]. By presenting one of these plenary lectures, Dickson joined the ranks of mathematicians like Felix Klein, Giuseppe Peano, Henri Poincaré, Émile Picard, Simon Newcomb, and Edmund Landau. Knowing he would have a distinguished international audience before him, Dickson must have carefully considered the topic of his lecture.

It came from the theory of numbers, that is, in Dickson's words, from "*the literature... I had been examining minutely in the preparation and publication of the first volumes of my* History of the Theory of Numbers.

I shall approach a few typical problems of the theory of numbers through the medium of other branches of mathematics" [Dickson 1921d, p. 41 or p. 579].[6] Thus Dickson's historical study of the theory of numbers inspired his International Congress address. He chose, in particular, to apply geometrical methods to the problem of finding all rational solutions of certain Diophantine equations and to make use of the theory of algebraic invariants in determining the integers for which a given binary cubic form equals a square [Dickson 1921d, pp. 42–46, 55–56 or pp. 580–584, 591–594]. He devoted the majority of his attention, however, to an application problem involving algebraic and hypercomplex numbers.

He opened briefly by sketching the development of his ideas with regard to this latter problem. "*While seeking interesting material which would illustrate this topic,*" he explained, "*I was led to the discovery of a very simple general method of finding explicit formulas which give all the integral solutions of homogeneous quadratic equations in several variables. For equations in four variables, the method makes use of some simple properties of integral algebraic numbers; while for equations in six variables, use is made of properties of integral quaternio*[*n*]*s*" [Dickson 1921d, p. 41 or p. 579]. With a "*method*" for solving these types of Diophantine equations which depended on properties of algebraic numbers and hypercomplex numbers, Dickson seized the opportunity to establish the relationship between the theories.

These introductory remarks also provided hints of Dickson's more general ideas regarding Diophantine analysis; in particular, he called for broader approaches to the study of Diophantine equations. He expressed this view candidly in the preface to the second volume of his *History.* "*Since there already exist too many papers on Diophantine analysis which give only special solutions,*" he declared, "*it is hoped that all devotees of this subject will in future refrain from publication until they obtain general theorems on the problem attacked if not a complete solution of it. Only in this way will the subject be able to retain its proper position by the*

[6] Here, knowingly or otherwise, Dickson continued in the spirit of the "*zero-th*" Chicago Congress of 1893 where Hermann Minkowski joined the "*branches*" of number theory and geometry. Specifically, Minkowski conveyed his now famous theorem concerning the existence of a point with integral coordinates (other than $(0,0,0)$) in closed, convex subsets in $\mathbb{R}^3$ that are symmetric about the origin. See [Minkowski 1896] and [Parshall & Rowe 1994, pp. 316–317].

side of other virile branches of mathematics" [Dickson 1920, p. xx]. From Dickson's perspective, then, the advancement of the subject depended on the determination of complete—as opposed to partial—solutions.

Dickson, though, especially desired integral solutions. He made this clear in his brief history of the development of these types of problems. "*Diophantine analysis,*" he began, "*was named after Diophantus, of the third century, who proposed many indeterminate problems in his arithmetic... He was content with a single numerical rational solution, although his problems usually have an infinitude of such solutions. Many later writers required solutions in integers (whole numbers), so that the term Diophantine analysis is used also in this altered sense. For the case of homogeneous equations, the two subjects coincide. But in the contrary case, the search for all integral solutions is more difficult than that for all rational solutions. In his first course in the theory of numbers, a student is surprised at the elaborate theory relating the equation which in analytic geometry represents a conic; but it is a real difficulty to pick out those points whose coordinates are integral*" [Dickson 1920, p. iii].

Dickson wrote these particular remarks in April of 1920. By the time of his Strasbourg address in September, however, his view of the difference between the determination of integral and rational solutions for homogeneous Diophantine equations had changed. Specifically, he "*note*[*d*] *the marked contrast between the problem of finding all the rational solutions and that of finding all the solutions in integers, in spite of the homogeneity of our equations*" [Dickson 1921d, p. 46 or p. 584]. In less than half a year, then, the problem of establishing the integral solutions for a homogeneous Diophantine equation went from one of "*coinciding*" with that of finding the rational solutions to the corresponding non-homogeneous equation to one standing in "*marked contrast*" to it. Perhaps Dickson intended for this comment to lure readers into the paper since it seems to suggest a departure from the more standard approaches to these types of problems.

Indeed, in Strasbourg,[7] Dickson's "*very simple general method*" did provide new insight into Diophantine Analysis when he determined the

[7] Six months later, in fact, he would generalize these results back in the United States at an AMS meeting [Dickson 1921c]. See below.

integral solutions for equations of the type

$$x_1^2 + x_2^2 + x_3^2 = x_4^2 \quad \text{and} \quad x_1^2 + \cdots + x_5^2 = x_6^2. \tag{1}$$

For each equation, he transposed one square and expressed the resulting difference of two squares as a product. In other words, he wrote the equations in this form

$$x^2 + y^2 = zw \quad \text{and} \quad x^2 + y^2 + z^2 + w^2 = sn. \tag{2a, b}$$

He thus reduced the Diophantine equations under consideration to expressions concerning the sums of two and four squares.

For the first of these two equations, Dickson found the rational solutions immediately.[8] He employed concepts he had discussed at length in the first volume of his *History*, namely, divisibility and primality. Specifically, he divided x and y by z (for $z \neq 0$) and expressed these rational numbers in terms of a common denominator, that is,

$$\frac{x}{z} = \frac{n}{m}, \quad \frac{y}{z} = \frac{r}{m}, \tag{3}$$

where m, n, r are integers without a common factor > 1. He then wrote the resulting equation as

$$\frac{w}{z} = \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = \frac{n^2 + r^2}{m^2}, \tag{4}$$

thus reducing the homogeneous equation (2a) to the corresponding nonhomogeneous equation in the customary way. By equating denominators and introducing a "*proportionality factor*"[9] ρ, he set $z = \rho m^2$, where ρ is rational, and gave the resulting values for the variables as

$$x = \rho mn, \quad y = \rho mr, \quad z = \rho m^2, \quad w = \rho(n^2 + r^2). \tag{5}$$

When $z = 0$, the rational solutions of (2a) have $x = y = 0$ and hence are given by (5) for $m = 0$ [Dickson 1921d, p. 47 or p. 585]. With this reduction

[8] In [Dickson 1920, pp. 305–323], Dickson primarily surveyed the results concerning the expression of integers (as opposed to rational numbers) as the sum of n squares.

[9] [Dickson 1921d, p. 47 or p. 585]. Dickson did not refer to ρ as the "*proportionality factor*" in his Strasbourg lecture. He used this term in two later publications, namely, [Dickson 1921c] and [Dickson 1929].

of the homogeneous Diophantine equation (2a) to the corresponding non-homogeneous equation (4), Dickson determined all rational solutions of (2a) for rational ρ's and integers m, n, and r with no common factor > 1. Ultimately, however, he desired integral solutions.

Typically, according to Dickson, "[*s*]*ome writers are in the habit of suppressing the proportionality factor* ρ [that is, restricting $\rho \in \mathbb{Q}$ to integral values] *and claiming without further examination that the resulting values give general solutions in integers*" [Dickson 1921c, p. 315 or p. 614]. But Dickson showed immediately the "*fallacy*" of this approach by considering the solution $x = 1$, $y = 3$, $z = 2$, $w = 5$ in the form of (5). Dividing y and z by x we see that $r = 3n$ and $m = 2n$ and n is a factor of r, m, and n. Since r, m and n have no common factor > 1, $n = \pm 1$ and $m = \pm 2$, $r = \pm 3$ and $\rho = \frac{1}{2}$. In particular, ρ is not an integer. In other words, Dickson recognized that finding all integral solutions to (2a) went beyond restricting the proportionality factor ρ to the integers.

As Dickson noted several months after the Strasbourg Congress,

"[*i*]*t has been regarded as self-evident by all writers,*[10] *who have mentioned the topic, that the problem of solving a non-homogeneous equation in rational numbers is equivalent to the problem of solving the corresponding homogeneous equation in integers... there is nothing wrong with the algebraic work* [deduced from the typical transformation from the homogeneous to the non-homogeneous equation]*, nor with the facts deduced. The fallacy lies in the failure to perceive that these facts do not warrant the conclusion that, in the converse case, we have shown how to find* all *integral solutions*" [Dickson 1921b, p. 313 or p. 612, my emphasis].

Indeed, Dickson devoted the majority of his proof to establishing this result. Moreover, he not only sought all integral solutions, but he also desired "*to obtain a formula which gives all the integral solutions of equation* (2a) *for integral values of the parameters*" [Dickson 1921d, p. 47 or p. 585].

Just how did Dickson find the integer solutions to $x^2 + y^2 = zw$? He abandoned traditional number-theoretic approaches using divisibility and

[10] Here, in a footnote, Dickson specifically cites Gauss as an example of one of these writers. In § 300 of the *Disquisitiones Arithmeticae*, Gauss claimed that "*it is thus clear that the solution of this equation* [$ax^2 + 2bxy + 2dx + 2ey + f = 0$] *by rational numbers is identical with the solution by integers of the equation* $at^2 + 2btu + cu^2 + 2dtv + 2euv + fv^2 = 0$" [Gauss 1966, p. 356].

primality and looked, instead, to the properties of the complex numbers. In particular, he made use of the unique factorization of the Gaussian integers and what he called "*the well-known fact that the norm* $x^2 + y^2$ *of the product*

$$x + yi = (m + qi)(n + ri) \tag{6}$$

of two complex numbers equals the product of their norms. Thus (2a) *has the solution*:

$$x = mn - rq, \quad y = mr + nq, \quad z = m^2 + q^2, \quad w = n^2 + r^2 \text{''} \tag{7}$$

[Dickson 1921d, p. 47 or 585]. Dickson assumed m, n, q, and r had no common factor > 1. Incidentally, this agrees with the rational solutions Dickson obtained earlier if you multiply these numbers by an arbitrary rational number ρ and set $q = 0$. Thus, relying on the two properties mentioned above, Dickson derived the expressions, up to an integral multiplier, for integer solutions of $x^2+y^2 = zw$. Still, one piece of Dickson's work remained undone.

He needed to prove that *all* integral solutions are obtained when an integer ρ is multiplied by integers of the form (7). To accomplish this, he considered the products of the numbers (7) by an irreducible multiplier s/p, where $s, p \in \mathbb{Z}$. He showed that when these products are integers, that is, when p divides the numbers in (7), then "*the quotients are expressible in the same form* (7) *with new integral parameters in place of m, n, q, r*" [Dickson 1921d, pp. 7–8]. This proof hinged on the fact that the complex integers follow the ordinary laws of divisibility. Thus, true to his stated goal, Dickson determined *all* the *integral* solutions of $x^2 + y^2 = zw$.

As promised in the introduction, the "*method*" used to find these solutions employed "*simple*" properties of integral algebraic numbers: norm, unique division, and (unique) factorization of complex integers. By calling his method "*new,*" Dickson seems to imply that his approach to this problem represented an innovative contribution to the study of Diophantine equations. As Dickson had noted earlier, Diophantus, for example, "*knew how to express the product of two sums of two squares as a sum of two squares in two ways,*" namely

$$(m^2 + q^2)(n^2 + r^2) = (mn \pm rq)^2 + (mr \mp nq)^2 \tag{8}$$

[Dickson 1920, p. vii].

Hence, although Dickson arrived at the solution to $zw = x^2 + y^2$ by using properties of integral complex numbers, he found a result he conceived of as dating back to Diophantus himself. But although Dickson called his method "*new,*" he did not explicitly mention that his approach to finding the solutions to $zw = x^2 + y^2$ was actually equivalent to the study of properties of $\mathbb{Z}[i]$, a method already familiar to Gauss. Perhaps by "*new*" Dickson meant the generalization of this method used in solving equations of the form $x_1^2 + \cdots + x_5^2 = x_6^2$ (see below). Or maybe by "*new*" he meant his emphasis on a *complete* determination of integral solutions. Interestingly, in his *History*, Dickson did not include advances made by Gauss using properties of the Gaussian integers in his discussion of the development of the "*two-square theorem*" [Dickson 1920, pp. 225–257]. It is possible, then, that he viewed his approach as unique since it relied less on properties of primes and congruences for specific cases in the style of his predecessors and more on arithmetic properties of algebraic numbers for general results.[11]

Whatever his reasoning, he immediately applied an analogous method using attributes of the integral quaternions to deduce all integral solutions to $x_1^2 + \cdots + x_5^2 = x_6^2$. For this part of his work, he relied on Adolf Hurwitz's arithmetic of quaternions [Hurwitz 1896]. Specifically, as Dickson put it, he needed "*a right-hand (as well as a left-hand) greatest common divisor of any two integral quaternions. Here a quaternion is called integral if its four coordinates are either integers or all halves of odd integers. The latter possibility would seem to present a difficulty in applying such an arithmetic of quaternions to the study of the integral solutions of this Diophantine equation; but we shall see that this difficulty is easily overcome...*" [Dickson 1921d, p. 51 or p. 589].[12] Dickson overcame this 2 in the denominator by restricting the definition of an integral quaternion to one with integer coordinates and defining a corresponding arithmetic which depended on odd norms.

[11] Dickson's generalization of these results in [Dickson 1921a, p. 354 or p. 620] seems to support this claim further. There, he asserts, "[w]*e shall see that the theory of algebraic numbers is admirably adapted to the complete solution of* N $[= x^2 - my^2$ or $x^2 + xy + ky^2] = zw$ *in integers.*"

[12] For the genesis of the definition of the concept of an integral quaternion, see [Fenster 1998].

Dickson fully developed these ideas the following year in his "Arithmetic of Quaternions" [Dickson 1921b]. As the title suggests, he considered not only the notion of an integral quaternion but also the attendant concepts of prime, greatest common divisor, and unique factorization of integral quaternions. He clearly spelled out the motivation behind his research in this direction when he wrote that "[*q*]*uaternions have recently been applied to the solution of several important problems in the theory of numbers. For this purpose it is necessary to make a choice of the quaternions which are to be called integral*" [Dickson 1921b, p. 225 or p. 397]. Thus, with the integral quaternions a necessary component in the search for solutions to certain Diophantine equations, Dickson found himself compelled to come to terms with the arithmetic of this specific algebra initially and with more general algebras later.[13]

His theory of the arithmetic of algebras hinged on the determination of a set of integral elements which led to an arithmetic analogous to that of the ordinary integers. As Dickson described it in the introduction to his celebrated text on the subject, *Algebras and their Arithmetics*,

"[*t*]*he chief purpose of this book is the development for the first time of a general theory of the arithmetics of algebras, which furnishes a direct generalization of the classic theory of algebraic numbers. The book should appeal not merely to those interested in either algebra or the theory of numbers, but also to those interested in the foundations of mathematics. Just as the final stage in the evolution of number was reached with the introduction of hypercomplex numbers (which make up a linear algebra), so also in arithmetic, which began with integers and was greatly enriched by the introduction of integral algebraic numbers, the final stage of its development is reached in the present new theory of arithmetics of linear algebras*" [Dickson 1923b, p. vii].

Inasmuch as his previous efforts to define a set of integral elements in an arbitrary algebra formed "*the final stage in the evolution of number,*" this book included the associated theory of arithmetic. Moreover, although the quaternions may have initially lured him into this subject, the generalization of the algebraic numbers represented a key component in the measurement of his theory's success.

As Dickson intended for his *Algebras and their Arithmetics* to reach a

[13] For the details of this work see [Fenster 1998, pp. 136–152].

wide audience, he devoted the first eight chapters to the development of the general theory of algebras. In particular, he called attention to Wedderburn's structure theorems of algebra which had previously remained "*somewhat overlooked*" [Birkhoff 1938, p. 287].[14] Moreover, Dickson's definition of a set of integral elements generally gained acceptance "*in the extensive German development of a unitary theory of ideals, by Emil Artin, Helmuth Hasse, the late Emmy Noether, B.L. van der Waerden, and others*" [Birkhoff 1938, p. 287]. Thus Dickson's work had strong ties to key mathematical ideas from the past, a broad presentation which brought many aspects of the general theory of algebras into focus, and links with the (future) work of the influential German algebraists. His impressive theory earned him the AAAS Prize (American Association for the Advancement of Science) in 1924 and the Cole Prize in 1928 for his book on the subject [Dickson 1923b], [Fenster 1998].

Conclusion

Dickson's 1920 reflections in his *History of the Theory of Numbers* on the general development of mathematics capture well his sense of mathematical progress. "[*C*]*onventional histories* [of mathematics]," he declared, "*take for granted that each fact has been discovered by a natural series of deduction from earlier facts and devote considerable space in the attempt to trace the sequence. But men experienced in research know that at least the germs of many important results are discovered by a sudden and mysterious intuition, perhaps the result of subconscious mental effort, even though such intuitions have to be subjected later to the sorting processes of the critical faculties*" [Dickson 1920, p. xx]. Although this comment expresses Dickson's general view of the evolution of mathematical results, it seems to apply equally well to his own specific path from the study of algebras to that of their arithmetics, since, *a priori*, there was no natural route from Diophantine equations to the arithmetic

[14] From Dickson's perspective, Wedderburn's structure theorems for rational algebras included: "[*e*]*very semi-simple algebra is either simple or is a direct sum of simple algebras, and conversely;* [the principal theorem:] *every algebra which is neither nilpotent nor semi-simple is the sum* $N + S$ *of its unique maximal nilpotent invariant sub-algebra* N *and a semi-simple sub-algbra* S" [Dickson 1928, p. 97 or p. 471] and "*every simple algebra* A *is a direct product of a simple matrix algebra and a division algebra* D*; this may be understood to mean that all elements of* A *can be expressed as matrices whose elements belong to* D" [Dickson 1924, p. 250 or p. 594].

of algebras. The joining of these seemingly disparate areas required some "*intuition*" which had grown from a near decade-long study of the history of the theory of numbers combined with close to twenty years of thoughts on algebras themselves. Moreover, the prestigious invitation to deliver a plenary lecture at the 1920 Strasbourg Congress may well have served to prompt Dickson's solidification of ideas on a variety of subjects, among them, the determination of solutions to certain Diophantine Equations.

Dickson's tersely written *History*, then, led to far more than self-satisfaction. At the time of its publication, and to some extent even now, it provided an invaluable source for those interested in number theory—professional and amateur alike—especially for those lacking adequate library facilities [Carmichael 1919, p. 397]. Moreover, as one reviewer asserted, it supported those "*who still believed in mathematics for mathematics' sake*" at a time when practical mathematical applications were held in high esteem [Lehmer 1919–1920, p. 125]. And, soon after its publication, for one young aspiring mathematician by the name of Richard Guy, Dickson's *History* proved "*better than ... the whole works of Shakespeare and heaven knows what else*" [Albers & Alexanderson 1993, p. 136]. Perhaps most importantly, however, the timely opportunity to deliver a plenary address at the 1920 International Congress planted the seeds for this altruistic historical study to grow into professional gold in the form of a theory of the arithmetic of algebras.

Thus this look at Dickson's *History* calls attention to the interplay between the history of mathematics and mathematical research. In particular, the differences in methodology between the practicing historian and the practicing mathematician come to the fore. No historian would have ever written Dickson's *History.* Dickson researched and wrote with the perspective—and aims—of a mathematician. Ultimately, it seems, his comprehensive study of number theory led him to isolate problems of interest and seek solutions of increasing generality. Not surprisingly, he spent the final decade of his mathematical career focused on establishing and generalizing a celebrated, unsolved number-theoretic problem [Dickson 1936]. This study, then, highlights not only how Dickson conceived of a historical study of number theory but also how he used the history of mathematics to inform his mathematical researches.

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