

**AN EXCITING NEW ARABIC VERSION  
OF EUCLID'S *ELEMENTS*:  
MS MUMBAI, MULLĀ FĪRŪZ R.I.6**

SONJA BRENTJES

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**ABSTRACT.** — This paper introduces an anonymous and undated Arabic version of Euclid's *Elements*. It tries to determine its relationship to the textual history of the Arabic *Elements* as known today. The value of the version, the paper argues, is its close relationship to the works of the first known translator of Euclid's *Elements* into Arabic, al-Ḥajjāj b. Yūsuf b. Maṭar, the light it sheds on philosophical debates surrounding the *Elements*, and the new textual basis (Books I to IX with some lacunae) it yields for the further study of the early history of Euclid's *Elements* in Arabic.

**RÉSUMÉ** (Une passionante nouvelle version arabe des *Éléments* d'Euclide : MS Mumbai, MULLĀ FĪRŪZ R.I.6)

Cet article présente une version arabe, anonyme et non datée des *Éléments* d'Euclide. Il vise à déterminer la relation de cette version à l'histoire textuelle des *Éléments* arabes telle qu'on la connaît aujourd'hui. Cette version est jugée intéressante pour le rapport étroit qu'elle entretient avec les ouvrages du premier traducteur connu des *Éléments* d'Euclide en arabe, pour les informations nouvelles qu'elle offre sur des débats philosophiques concernant les *Éléments* et finalement pour la base textuelle nouvelle qu'elle met à notre disposition pour des études plus approfondies sur la première période de l'histoire des *Éléments* d'Euclide en arabe.

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S. BRENTJES, independent scholar.

Courrier électronique : [sbrentjes@hotmail.com](mailto:sbrentjes@hotmail.com)

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## 1. INTRODUCTION

The textual history of Euclid's *Elements* in Arabic is multifaceted and far from being deciphered in a convincing manner. Four major factors have caused this unsatisfying situation. The first of these four factors is the complexity of the texts found in the preserved manuscripts as well as of the stories narrated in medieval Arabic sources about this history. A second factor is the scarcity of reliably ascribed and dated textual witnesses of major components of this history. A third factor is the focus of modern researchers on mathematical aspects of Euclid's work and their fate in the hands of scholars from Islamic societies. The final factor is the lack of interest among modern researchers for the study of philological and visual elements of the text and its numerous versions and variants. Information stored in medieval sources was and is often taken at face value. The order and content of definitions, postulates, axioms, and theorems as well as their proofs attracted much more solid attention than the analysis of any given book of the *Elements* in its entirety. The philological properties that may lead to identifying different translators, editors, or users and the variances between the diagrams that may highlight the functions attributed to visual knowledge as well as the relationship between individual manuscripts are most often considered at best of secondary importance to the historical project at large. Hence, several unfounded claims about the origin of entire manuscripts, certain theorems and definitions as well as individual technical terms have been made in the past.

The manuscript, which I will introduce in this paper, possesses strikingly peculiar features that allow excluding a set of fragments characterized by shared technical terms from the primary transmission of Euclid's *Elements* in Arabic. The primary transmission of Euclid's *Elements* designates all texts that can be proven to be translations into Arabic of a Greek or Syriac version of Euclid's work. Due to the broad range of skills needed in the process of translating Greek and Syriac scientific texts into Arabic in the eighth and ninth centuries, the translations were often submitted to proofreading or other procedures of correction by a colleague. Furthermore, due to various other factors such as the vivacious interest in translated scientific texts in Baghdad, the capital of the Abbasid Caliphate and center of the translation efforts, the potential of a scholarly career at court or the continuously changing accuracy, efficacy and range of scientific terminology, translations quickly became obsolete or at least old-fashioned. As a result, they were either replaced by new translations produced by younger scholars or by editions. The latter came either from the pen of the original translator(s) or were produced by scholars interested in the discipline and the subject matter of the text. In respect to these various

follow-ups of any given translation, new translations as well as editions by a translator will also be understood as components of the primary transmission of Euclid's *Elements* in Arabic. Editions, epitomes, paraphrases, or commentaries by scholars not directly involved in the production of a translation will be referred to as components of the secondary transmission of Euclid's *Elements*. The secondary transmission of Euclid's *Elements* in Arabic also comprises translations into other languages such as Latin, Syriac, Persian, or Sanskrit.

Several scholars contributed from the eighth to the tenth centuries to the emergence of the primary transmission of Euclid's *Elements* in Arabic. The most important names to be mentioned here, since they will be referred to in my analysis of MS Mumbai, Mullā Fīrūz R.I.6 (called from now on: MS Mumbai), are al-Ḥajjāj b. Yūsuf b. Maṭar (fl. ca. 786–833), Iṣḥāq b. Ḥunayn (d. 911), and Thābit b. Qurra (d. 901). Several medieval Arabic sources, among them the *Kitāb al-fihrist* compiled by Ibn al-Nadīm (d. 995), a bookseller and member of intellectual circles in Baghdad in the second half of the tenth century, and the preface to one of the two extant Arabic manuscripts of Abū l-'Abbās al-Nayrīzī's (d. ca. 922) commented edition of the *Elements* report that al-Ḥajjāj b. Yūsuf b. Maṭar translated the *Elements* either for the Abbasid caliph al-Hārūn al-Rashīd (r. 786–809) or on order of his vizier Yaḥyā b. Khālīd al-Barmakī (ex. 805). He is also credited with having produced either a new translation or a substantial edition of his old translation almost a quarter of a century later for the then reigning caliph al-Ma'mūn (r. 813–833). Ibn al-Nadīm claimed that this new translation superseded the first one. The author of the preface to al-Nayrīzī's work characterized in contrast the edition as a version that cut out superfluties, corrected errors, filled gaps, and improved upon the translation's language. As I have argued in other papers, the extant fragments that can be connected to al-Ḥajjāj's work suggest thinking of his second version as an edition rather than as a fresh translation [Brentjes 1994; Brentjes 1996]. All textual fragments that can be connected to al-Ḥajjāj's work, or at least said with some confidence to have been derived from it, will be labeled as members or derivatives of the Ḥajjāj tradition of the *Elements*. On the basis of Ibn al-Sarī's (d. 1153) testimony, Djebbar has proposed to consider one such fragment as a remnant of al-Ḥajjāj's original translation [Djebbar 1996, p. 103]. This fragment possesses a particular terminology, namely *talbīn* = the making of bricks, which it uses for describing squares and rectangles. This terminology altered the Greek way of speaking of these two types of figures as something being above a line into something that was made like a brick (of a size)  $a$  times  $b$  or  $a$  times itself. The term *talbīn*, a verbal noun of the second root of the verb *labana*, is – as far as I know – not attested in dictionaries

of classical Arabic. This fact and its use as if it signified the result of a process, not the process itself, i.e., a brick rather than the making of a brick, implies an origin in a context of translators whose mother tongue was not Arabic. Perhaps it was a word used by the first translator of Nikomachos of Gerasa's *Introduction into Arithmetic*, who worked in the early ninth century for caliph al-Ma'mūn's general Ṭāhir b. Ḥusayn (d. 822). Djebbar supported his identification of such fragments with al-Ḥajjāj's original translation by pointing to the practical connotations of the term and its similarities to other terms of an apparently analogous practical character which are known from other fragments ascribed to al-Ḥajjāj's work [Djebbar 1996, pp. 98–104]. I have argued that the fragments using the terminology of bricks show strong features of change and hence cannot be accepted as a remainder of al-Ḥajjāj's translation without further arguments and other textual witnesses [Brentjes 1994, pp. 84–91]. The text contained in MS Mumbai is such a new witness. This fact constitutes one aspect of its importance for the study of the textual history of Euclid's *Elements* in Arabic. I will show that MS Mumbai speaks against the origin of the *talbīn* terminology in al-Ḥajjāj's translation. Rather, it possesses features that point to an origin of this terminology in the secondary transmission of the *Elements*. These specific features linking an apparently practical terminology to the secondary transmission of the Arabic *Elements* and in particular to philosophical debates about the ontological and epistemological status of philosophical and mathematical disciplines are another aspect that makes this new textual witness of the *Elements* in Arabic exciting. They underline that interpretations of texts without investigations of their contexts tend to reflect more our own beliefs than those of the historical actors.

Almost half a century after al-Ḥajjāj's second version, Ishāq b. Ḥunayn translated Euclid's *Elements* anew. He gave his text to the mathematician and translator Thābit b. Qurra, who edited it. It is not clear what kind of changes were involved in this process of editing. As a student of his highly skilled father Ḥunayn b. Ishāq (d. 867), who had translated many Greek medical works and is generally hailed as the best translator of the ninth century, Ishāq b. Ḥunayn had an excellent training as a translator from Greek or Syriac into Arabic. Hence, it is not very likely that Thābit b. Qurra interfered much in his colleague's Arabic style and choice of words. Indeed, extant manuscripts of the first two books of the Arabic *Elements* ascribed to Thābit b. Qurra show that this assumption may be correct. The language in these manuscripts namely features an undeniable and substantial influence of Greek syntax. The neglect of proper Arabic syntax is most likely not an expression of Ishāq b. Ḥunayn's lack of knowledge, but the result of a conscious adherence to Greek style

based on beliefs about what constituted a good translation. Thus, Thābit b. Qurra may have focused mainly on improving the mathematical content of the text produced by Ishāq b. Hunayn. According to notes in extant manuscripts, he did this by adding alternative proofs and occasionally an additional theorem.

In the first two books of the extant manuscripts containing texts ascribed to Ishāq b. Hunayn and Thābit b. Qurra such as MS Tehran, Malik 3586, the latter's additions show philological features that set them apart from Ishāq b. Hunayn's translated text. Thābit b. Qurra apparently did not share his colleague's view on what constituted a good translation. He did not try to preserve Greek syntax in Arabic nor did he use the same technical terminology as Ishāq b. Hunayn.

The subsequent books present a much more complicated appearance. The extant manuscripts contain different variants, possibly even versions, of texts that are ascribed partly to Thābit b. Qurra and partly to al-Ḥajjāj b. Yūsuf b. Maṭar. Their philological features are much less pure than in the first two books. In particular, they lose their clear remnants of Greek syntax. Furthermore, the terminology shifts considerably and loses in some books its previous stability. Several, often conflicting judgments have been offered to the effect that certain of these manuscripts carry a text of the Ḥajjāj tradition, while others contain the text translated by Ishāq b. Hunayn and corrected by Thābit b. Qurra (abbreviated from now on as: Ishāq-Thābit tradition). Without a meticulous analysis of the features of these variants or versions – mathematical, philological, and visual –, however, no reliable results can be achieved.

MS Mumbai deserves our particular attention because of several peculiar features. Including the two aspects I already mentioned earlier, five features constitute the specific value of this manuscript for the further study of the textual history of the Arabic *Elements*. First, the manuscript contains a text that includes in its margins, as interlinear glosses, and as interpolations variants ascribed to Thābit b. Qurra and al-Nayrīzī as well as anonymous comments. This relationship indicates that the main text of this manuscript was not believed to originate from either of the two scholars. Second, parts of the anonymous comments are closely related to fragments, often ascribed to the philosopher and scientist Ya'qūb b. Ishāq al-Kindī (d. ca. 873). Third, the main text, not the glosses, comments, and interpolations contain some technical termini, among them the one mentioned earlier (*talbīn*), that are either linked in the sources to the work of al-Ḥajjāj or are known from translations of philosophical or mathematical texts made in the eighth and early ninth centuries such as *al-ṭīna* (matter). Fourth, parts of the anonymous comments explain the shift in technical terminology as the result of a philosophical

conviction according to which numbers possess a higher ontological status than geometrical objects. Hence, the earlier discussed assumption of Djebbar that *talbīn* was a technical term taken from everyday-life language and connected to other such terms ascribed explicitly to al-Ḥajjāj needs to be abandoned. Fifth, the manuscript contains large parts of the first nine books of the *Elements*. It is thus the longest extant coherent text that works with this particular terminology as well as other terms affiliated with the Ḥajjāj tradition, without being, as I will argue, a genuine Ḥajjāj text. Hence, the manuscript contains an unknown annotated edition of one of the two versions of al-Ḥajjāj. It originated without doubt before the twelfth century, when other Arabic texts confirm the use of *talbīn* and Ibn al-Sarī labeled a theorem using this terminology as the translation of al-Ḥajjāj [Djebbar 1996, p. 103]. The text's derivation from a Ḥajjājian work, its possible relationship to al-Kindī, and the use of philosophical principles suggest placing its origin in the first half of the ninth century. As I have already stated philological features of the text support such an early dating.

My paper focuses on the question as to who may have been the author/s of the core text in MS Mumbai, i.e., the text of the *Elements* proper, and hence on the date of this version.

## 2. WHO WAS OR WERE THE AUTHOR/S OF THE EDITION IN MS MUMBAI?

MS Mumbai, Mullā Fīrūz, R.I.6 consists of Books I to IX and breaks off in the proof of theorem IX, 29. As a rule, it omits the conclusion at the end of a proof. There are three major lacunae – the first in Book II (theorems II, 5-6 and most of theorem 7), the second at the beginning of Book III (the definitions, theorems III, 1-6, and the beginning of III,7), and the third at the end of Book VI and the beginning of Book VII (most of VI,31; VI,32; definitions Book VII; VII,1; beginning of VII,2). The folios containing theorems I, 18-25 have been misplaced. The version contains two introductions and a concluding summary in Book I referring back to the first introduction as well as an introduction and a concluding summary in Book II. The introductions to Book I are known from various other sources. The concluding summaries of Books I and II and the introduction to Book II are unique, i.e., not known from any other source. Books III–IX include 25 major glosses, mostly taken from texts of the Ishāq-Thābit tradition and copies of al-Nayrīzī's commentary. In one case, a scribe or editor may also have worked directly with the Arabic translation of Heron's commentary rather than with its quotation by al-Nayrīzī. Several philological peculiarities, in addition to the two I have

already mentioned, characterize the core text found in MS Mumbai. This text uses the language of bricks until Book VII. In Books VII–IX, however, it stops doing so. This may be a reflection of the philosophical reason for introducing this language in order to transform geometrical objects into rational objects.<sup>1</sup> Hence, arithmetical objects, the subject matter of Books VII–IX, may not have drawn the attention of the philosophically minded editor, even when they were talked about in geometrical terms as it happens repeatedly in these three books. In these three number theoretical books, MS Mumbai also uses terms well-known from other fragments linked to the Ḥajjāj tradition such as *bāyana*, *tabāyana*, and their derivatives for relative prime (numbers) and *mushtarak* for not relative prime (numbers) [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 61b-64a]. *Jidhr* (root), squares, and cubes appear in theorem VII,27, but have no direct equivalent in Heiberg's Greek text [MS Mumbai, Mullā Fīrūz, R.I.6, f. 62b,17]. Finally, starting from theorem VII,33, MS Mumbai uses the cumbersome introductory formula for problems also well-known from certain other versions of the Arabic *Elements*, for instance al-Nayrīzī's commentary:

“*nurīdu an nubayyina kayfa ...*” [we wish to explain how ...] [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 63b,27 (theorem VII,33); 64a,19 (VII,34); 65a,1 (VII,36); 65b,10 (VII,39)].

The late usage of this particular formula within MS Mumbai implies that it is an interpolation.

Certain of these lexical peculiarities as well as some structural differences to Heiberg's Greek text and extant manuscripts of the Arabic primary transmission as well as basic structural similarities with versions of the Arabic and Arab-Latin secondary transmission linked to or explicitly ascribed to al-Ḥajjāj leave no doubt that the edition in MS Mumbai is an indirect witness of the Ḥajjāj tradition.<sup>2</sup>

This result leads to the question as to whether some part of this edition can be regarded as a direct witness of the Ḥajjāj tradition. On the level of the text as presented in MS Mumbai, this question can be answered negatively. The text as a whole contains too many additions and interpolations. On the level of a text entirely cleansed from these additions and interpolations, the answer is less unequivocal. The analysis of the

<sup>1</sup> The meaning of these terms and their philosophical background will be discussed in subsection 2.4.

<sup>2</sup> Examples are the proofs of theorems III, 24, 32, 35, 36 where specific cases are considered or the order of a good number of theorems in Book VI which in numbers of Heiberg's edition is as follows: VI, 13, 11, 9, 10, 14-17, 19, 20, 18, 21, 22, 24, 26, 23, 25, 27-30, 32, 31. This order agrees with the one found in Adelard of Bath's translation [Busard 1983, pp. 174–95].

possible interpolations and additions suggests asking a further question, namely whether MS Mumbai's text including the introductions and the comments linked to them, but without the glosses and marginal notes, was the work of one single author. Finally, beyond the question of the Arabic textual history of the *Elements*, MS Mumbai offers new evidence for discussing the Greek history of the Euclidean work.

I will discuss the three points in six steps. I will start with the second question and devote to it steps one to four. Then I will turn to the first question. Finally I will deal with the third question. First, I will show that the first introduction and the concluding remarks to Book I have been composed by an author who had seen a version of the *Elements* from the Iṣḥāq-Thābit tradition and the translation of Books XIV and XV traditionally ascribed to Qusṭā b. Lūqā (ninth century). I will argue that in all likelihood this author was not al-Kindī. Second, I will discuss the text of MS Mumbai freed from all glosses and marginal notes, as well as the first introduction and the concluding remarks to Book I. I will argue that the author of this reduced text cannot have been al-Kindī or his contemporary al-'Abbas b. Sa'īd al-Jawharī (ca. 800–860) who also produced an edition of the *Elements*. Third, I will offer arguments for why it is highly unlikely that the language of bricks was the language of al-Ḥajjāj's translation of the *Elements* as claimed by the twelfth-century mathematician Ibn al-Sarī [Djebbar 1996, p. 103]. Fourth, I will discuss the additions to Book II and some glosses to Book V. I will argue that this material implies that the author of the additions to Book II as well as the author of the core text of MS Mumbai's version lived in the early ninth century. Fifth, I will discuss which kind of reduced form of MS Mumbai's version can perhaps be regarded as a direct witness to al-Ḥajjāj's edition. In a final subsection, I will present examples where MS Mumbai seems to preserve an older stratum of the Greek *Elements* than the one known from Heiberg's edition and from the extant manuscripts of the Arabic primary transmission.

### **2.1. *The philological and textual environment of the first introduction to Book I in MS Mumbai and its concluding paragraph***

The terminology of the first introduction to Book I, while clearly linked to the terminology used in MS Mumbai's core text of the *Elements*, shows a contradictory picture. It contains on the one hand philological features that support its ascription to an early period of Arabic scientific language.<sup>3</sup> On the other hand, in the definitions of Books I and VII, the first

<sup>3</sup> Such features are the use of the feminine rather than the masculine form of *muthallatha* (triangle), *murabba'a* (quadrangle or square), and *munḥarifa* (trapeze), the use of verbal

introduction uses a terminology that is characteristic of the Ishāq-Thābit tradition [MS Paris, BNF, A 2457, ff. 23b-31a]. A particularly rare piece of technical terminology in the first introduction to Book I in MS Mumbai is *al-izām*, which is used as a parallelism to *al-aqdār*, both representing the Greek μέγεθοι (magnitudes). It was used by Abū 'Uthmān al-Dimashqī (d. 910) in his translation of Pappos' commentary on Book X of the *Elements* [MS Paris, BNF, A 2457, f. 2a, 25-2b,3].

Furthermore, the first introduction to Book I in MS Mumbai talks of all fifteen books of the *Elements*, while the historian Aḥmad b. Wādiḥ al-Ya'qūbī spoke of thirteen books of the *Elements* only in his book on world history [al-Ya'qūbī 1379h/1960, vol. 1, p. 120]. Al-Ya'qūbī's statement shows that at least one of the two versions produced by al-Ḥajjāj incorporated only the genuine Euclidean books. Hence, the first introduction to Book I in MS Mumbai was presumably written only after Qusṭā b. Lūqā had translated the additional two books XIV and XV. The presence of *talbīn* in the first introduction as preserved in MS Paris, BNF, A 2457 proves that the language of bricks had been introduced by this time into one of the Arabic versions of the *Elements*. This observation implies that the language of bricks was invented by an author of the earlier ninth century, since Qusṭā b. Lūqā was probably born around 820.

Aside from the two introductions, Book I in MS Mumbai contains two more editorial elements – its concluding paragraph and the comments to its postulates, axioms, and definitions. The concluding paragraph, being tightly linked to the first introduction, quotes also the enunciation of theorem II,14 in a different formulation than the one found in MS Mumbai's core text of Book II. Thus, the concluding paragraph presents a picture similar to that of the first introduction. The linguistic differences that these two textual units display in respect to MS Mumbai's core text are meaningful in most instances only in an Arabic environment that had already produced several alternative translations and modifications for one and the same Greek mathematical term. Hence, the author of the first introduction and of the concluding remarks to Book I worked either with a contaminated Arabic version of the *Elements* composed from elements taken from three types of sources: a version of the *Elements* using the

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nouns rather than particles such as *al-tasāwī* instead of *al-mutasāwī* (being equal to each other), or the use of the plural form for a feminine adjective accompanying a feminine noun in plural form such as *murabba'āt mutasāwīyyāt* (squares, which are equal to each other) [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 1b,8; 2a,21-25; 3b,13-14].

language of bricks, the Ishāq-Thābit tradition, and Qusṭā b. Lūqā's translation of Books XIV and XV as well as his translation of Pappos' commentary on Book X. The other option is that this author worked with several Arabic versions of the *Elements* and commentaries.

This philologically and methodologically complex environment of Book I in MS Mumbai suggests the question: are the comments on its definitions, postulates, and axioms linked to its first introduction and concluding remarks or are they from a different author? In some sense, the comments, the first introduction, and the concluding remarks are linked on a conceptual and philological level. Both texts talk about genus and species of geometrical objects. The linkage of this kind of exposition to Aristotelian logic is obvious. The first introduction opens, for instance, with the claim that Euclid's purpose was to explain the properties of the quantity, its genera, and its species. The properties are defined as equality, inequality, and what followed from the two. The genera are described as continuous and discrete. And the species are the line, the plane, the solid in the genus of the continuous and the number in the genus of the discrete [MS Mumbai, Mullā Fīrūz, R.I.6, f. 1b,2-6]. As for the comments on the definitions, the discussion of the genera of the quantities starts in respect to definitions I,2 and 3. Except for the term used for quantity (*al-kammiya* in the first introduction; *al-miqdār* in the definitions), the technical terminology used in this discussion does not differ between the two kinds of textual units. The comment states that the line is the first genus of the quantity. Then it justifies why the line is defined as a length only in the *Elements*: "since it is a section of the plane and divided by the point" [MS Mumbai, Mullā Fīrūz, R.I.6, f. 2b,21-22].

This justification indicates that despite the previously elucidated similarities there is an undeniable difference in the philosophical stance between the first introduction and the concluding remarks of Book I in MS Mumbai and the comments on its definitions, postulates, and axioms. The comment's atomistic outlook reappears in the comment on definitions I,6 and 7. As in the comment on definitions I,2 and 3, the defined object is described as deriving from an object defined later as well as from an object defined earlier, i.e., the plane derives both from the solid and the line: "since it is the section of the solids and it is divided by the line" [MS Mumbai, Mullā Fīrūz, R.I.6, f. 2b,25-26]. The author of the first introduction took a different approach. He claims that the earlier defined objects exist only because of the existence of later defined objects, i.e., the plane exists only because of the solid. No references to atomistic concepts are made by this author [MS Mumbai, Mullā Fīrūz, R.I.6, f. 1b,9-11].

The conflict between the two approaches is of such a nature that the assumption of two different authors seems to be more plausible. A possible author of the first introduction and the concluding remarks in Book I is Abū Naṣr al-Fārābī (870–950/51). Al-Fārābī's view about the genesis of the geometrical objects agrees with that summarized in these two textual units. He also wrote a commentary on Books I and V of the *Elements*. This extant commentary differs, however, completely from what is found in MS Mumbai [Shamsi 1984, pp. 59–60]. Hence, it is more likely that another Aristotelian philosopher or a commentator with Aristotelian leanings wrote the first introduction and the concluding remarks in Book I of MS Mumbai.

The author of the comments on the definitions is equally difficult to pin down. In the middle of the eighth century, Ibn al-Muqaffa' (ex. ca. 760) had already offered in his exposition of logic an atomistic perspective with regard to point, line, and solid in his discussion of Aristotle's *Categories* [Ibn al-Muqaffa' 1978, p. 24]. Since he used, however, a completely different technical terminology, it is highly unlikely that he wrote the comments on the definitions in Book I of MS Mumbai. The conceptual linkage of those comments to Ibn al-Muqaffa' could suggest, however, that their author and the author of the introduction and the concluding remarks to Book II in MS Mumbai was one and the same person, because the introduction to Book II is terminologically linked to Ibn al-Muqaffa''s *Logic*. Ibn al-Muqaffa' subscribed to a peculiar interpretation of Aristotelian logic, which is not only expressed in his atomistic stance, but also in his inclusion of time, place, line, plane, and solid as species of the genus *al-manẓūm* (the orderly, i.e., the continuous) together with calculation and speech as species of the genus *al-maqṭū'* (the cut off, i.e., discrete) into the category of number [Ibn al-Muqaffa' 1978, p. 12]. His technical language is characterized by terms, which resurface – sometimes with a radically different connotation – in Arabic extracts of the *Elements* linked to the Ḥajjāj tradition.<sup>4</sup> The version of MS Mumbai also contains terms found in Ibn al-Muqaffa''s *Logic* and fragments of the Ḥajjāj tradition of the *Elements*.<sup>5</sup> This shared terminological and conceptual background suggests that the author of the comments on

<sup>4</sup> Examples are *aṣamm* for incommensurable (Book X), *juththa* for solid (Book XI), and *ghilaz* for thickness (Book XI) [Ibn al-Muqaffa' 1978, p. 12].

<sup>5</sup> Examples are *lahiqa bi* for being connected with (literally: to stick to each other, to glue together) and *mutabāyin* for being separated from each other or relatively prime (literally: differing, dissimilar, unlike, varying) [Ibn al-Muqaffa' 1978, p. 12; MS Mumbai, Mullā Fīrūz, R.I. 6, f. 3b, 21; Ibn Sīnā 1976, p. 19].

the definitions in Book I and of the introduction and concluding remarks to Book II of MS Mumbai lived early in the ninth century.

## 2.2. *Did al-Kindī or al-Jawharī compile the edition in MS Mumbai?*

The relationship between the second introduction to Book I in MS Mumbai and the extant fragment of the analogous introduction to al-Kindī's edition of the *Elements* suggests that al-Kindī's edition differed from the one preserved in MS Mumbai. Al-Kindī's terminology in this fragment shows clear signs of being edited on the basis of a particular group of the extant variants of MS Mumbai's second introduction.<sup>6</sup> Thus, it is probable that the core text of al-Kindī's edition, which is not known to be extant, also differed from the core text of MS Mumbai. The commentary on the *Elements* composed by the tenth-century scholar Aḥmad al-Karābīsī lends support to this assumption. It contains a variant of definition I,8 ascribed to al-Kindī. This variant agrees with MS Mumbai in several major terminological and conceptual points, but also deviates from it in more than one way.

MS Mumbai, I,8:

“And the plane (*musattaha*) angle is the touching (*tamāss*) of two lines and their inclination (*inḥidābuhuma*) on a surface (*sath*), whereas their connection with each other (*ittiṣāluhuma*) is in a non-straight extension” [MS Mumbai, Mullā Fīrūz, R.I.6, f. 2b,8].<sup>7</sup>

Al-Kindī according to al-Karābīsī:

“And Ya'qūb b. Ishāq al-Kindī defined the plane (*musattaha*) angle by saying: it is the touching (*tamāss*) of two lines in a non-straight extension at a point, (which) encompass (*yuhīlān*) a surface (*sath*)” [Brentjes 2000, p. 47].<sup>8</sup>

A search for other Arabic variants of this definition shows that such a behavior is by no means rare. Other authors kept other elements characteristic for the formulation in MS Mumbai. An example for such a further variant is the definition in Ibn Sīnā's paraphrase:

<sup>6</sup> This particular group consists of five sources: al-Ya'qūbī's summary, the proper text of the *Elements* in al-Nayrīzī's commentary, two manuscripts of the Arabic primary transmission of the *Elements*, i.e., MS Oxford, Bodleian Library, Thurston 11 and MS Rabāt, al-Maktaba al-Malikiyya, al-Khizāna al-Ḥasaniyya 1101, and Gerard of Cremona's Latin translation of a highly contaminated text of the Arabic primary transmission.

<sup>7</sup> والزاوية المسطحة هي تماس خطين وانحدابهما على سطح واتصالهما على غير استقامة

<sup>8</sup> فقال اسحق بن يعقوب الكندي الزاوية المسطحة هي تماس خطين على غير استقامة على نقطة واحدة يحيطان بالسطح

“And the plane (*musatṭaha*) angle is that which two lines encompass (*yuhṭṭu*) (when) linked (*muttaṣilān*) not in straight (extension and) inclined (*mutaḥadibān*) in a plane (*sath*)” [Ibn Sīnā 1976, p. 17].<sup>9</sup>

Additionally, two further main versions of definition I,8 can be found in Arabic texts of the *Elements*. Both of them show no resemblance to the one in MS Mumbai. One group is characterized by its use of *zāwiya musatṭaha* for plane angle, *mayl* for inclination, *mutalāqī* (meeting) for being connected with each other, and *muttaṣil ‘alā ghayr istiḳāma* (linked in a not-straight extension) for not being put as one straight line. The other group employs *zāwiya basiṭa*, *inḥirāf*, and *muttaṣil ‘alā ghayr istiḳāma* for the cases one, two, and four, while case three is missing altogether. The first group is represented, for instance, by definition I,8 in al-Karābīsī’s and al-Nayrīzī’s commentaries [Brentjes 2000, p. 61; MS Qom, Kitābkhāne-yi ‘Āmm-i Ayātollāh-i ‘Uḏmat-i Mar’ashī 6526, f. 2b,15-18]. Manuscripts of the Arabic primary transmission contain the second group [MS Tehran, Malik 3586, f. 249b,9-10].

The phrasing of definition I,8 in MS Mumbai does not resemble any of these two groups. It shares, however, with the first group its equivalence with the Greek definition as edited by Heiberg, since it contains translations of all central components of the Greek definition. In contrast, the phrasing ascribed to al-Kindī omits one of the Greek components (*ἄπτομένοσ*), as does the Arabic primary transmission. Additionally, al-Kindī cancelled a word for inclination (*κλίσις*) and introduced with *yuhṭṭān bi’l-sath* (encompassing a plane) a new concept. He borrowed it most likely from (an Arabic translation of) a Greek commentary or a compilation of Greek scholia on the *Elements*, since this concept is also found in an alternative definition ascribed by al-Nayrīzī and al-Karābīsī indirectly to Greek authors (see [Arnzen 2002, pp. XXVII–XXIX]; [Brentjes 2000, pp. 36, 38]).

Further support for the view that al-Kindī was not the author of the ancestor of MS Mumbai can be found in two glosses to postulate I,1 and axiom I,1, which are preserved in the Leiden manuscript of al-Nayrīzī’s commentary. They are both attributed explicitly to al-Kindī and formulate the two items in a different technical language than did the ancestor of MS Mumbai [Codex Leidensis 399.1, p. 16, footnote 1 and p. 28, footnote 1]. Postulate I,1 in MS Mumbai resembles closely the formulation in manuscripts of the Arabic primary transmission, while axiom I,1 in MS Mumbai coincides with the form given as a gloss in the Leiden

<sup>9</sup> والزواوية المسطحة هي التي يحيط بها خطان متصلان لا على الاستقامة متحديان على سطح

manuscript of al-Nayrīzī's commentary.<sup>10</sup> The highly contaminated and irregular character of all primary Arabic and Arab-Latin versions of the *Elements* makes it impossible to draw any reliable conclusion from the overwhelming agreement between MS Mumbai and the manuscripts of the Arabic primary transmission in the case of postulate I,1. No possible interpretation can truly be excluded, i.e., the form of the postulate can come from either of the two Arabic traditions.

After having argued against al-Kindī as the possible author of MS Mumbai's ancestor, what can be said about al-'Abbas b. Sa'īd al-Jawharī? His *Iṣlāḥ* is almost as completely lost as al-Kindī's, i.e., we only possess a brief description and a short extract in one of the works by Naṣīr al-Dīn al-Ṭūsī (1201–1274). Ṭūsī characterized al-Jawharī's edition as follows:

“As for al-Jawharī, may God show mercy upon him, he has an edition of the book of the *Elements* to which he has added to the beginnings of each art (*fann*) introductions/lemmata (*muqaddimāt*) and *termini technici* and to the theorems approximately fifty (additional) theorems” [Jaouiche 1988, p. 171].<sup>11</sup>

MS Mumbai does not contain fifty additional theorems nor lemmata and explanations of technical terms. Except for the first two books, there are no introductions either. Hence, it is not very likely that MS Mumbai's ancestor was composed by al-Jawharī.

### 2.3. *The language of bricks versus Euclid's geometrical language*

As reported earlier, the mathematician Ibn al-Sarī claimed in the twelfth century that al-Ḥajjāj used the language of bricks in his translation. This claim is highly implausible when confronted with the various preserved witnesses for those theorems in Books I, II, III, and VI, which use the language of bricks in MS Mumbai as well as other extant fragments of the *Elements* related to the Ḥajjāj tradition such as the one extant in MS Paris, BNF, Suppl. Persan 169. Moreover, the *Rasā'il ikhwān al-ṣafā'*, an encyclopedic work of the late tenth century, preserve in their second chapter on geometry a purely geometrical variant of theorems II,12 and 13 that is closely connected to the formulation of these theorems in MS Mumbai. To illustrate the similarity it suffices to quote the enunciation of II,12:

<sup>10</sup> Postulate I,1 in MS Mumbai agrees completely with the formulation found in MSS Rabāṭ, al-Khizāna al-Ḥasaniyya 53, p. 47, 13; Rabāṭ, al-Khizāna Ḥasaniyya 1101, f. 2; Madrid, Escorial A 907, f. 2a, 4-5. It agrees except for one or two words with all the other manuscripts available to me.

<sup>11</sup> واما الجوهر ي رحمه الله فله اصلاح لكتاب الاصول وقد زاد في مباديء كل فن  
مقدمات ومصطلحات وفي اشكال لكتاب قريبا من خمسين شكلا

*Rasā'il ikhwān al-ṣafā'*:

“A property of the obtuse-angular triangle is that the square of the chord is greater than the square of the two sides by twice the quantity of the tetragon of one of the two sides with that, which is outside of it until the foot-point of the perpendicular” [*Rasā'il ikhwān al-ṣafā'* 1957, vol. 1, p. 106].<sup>12</sup>

## MS Mumbai:

“[For] each obtuse-angular triangle: if one of the two sides encompassing it is drawn, whichever of the two it may be, until the foot-point of the perpendicular, then the brick of its chord times itself is larger than the brick of each one times itself (and taken) together by twice the brick of the side, which is drawn from it until the foot-point of the perpendicular” [MS Mumbai, Mullā Fīrūz, R.I.6, f. 17a,1-8].<sup>13</sup>

## MS Paris, BNF, Suppl. Persan 169:

“[For] each obtuse-angular triangle: if from its obtuse angle one of the two sides encompassing it, whichever it may be, is drawn until the foot-point of the perpendicular on which it (i.e., the perpendicular) falls outside the triangle, then the brick of the chord of the obtuse angle times itself is greater than the brick of the two sides encompassing it, each one times itself (and taken together), by twice the brick of the side drawn from it times that, which was drawn from it until the foot-point of the perpendicular” [Brentjes 1994, p. 64].<sup>14</sup>

The juxtaposition of the three texts shows that all three variants are closely interrelated, but possess at the same time indisputable signs of interference either by editors or by scribes. None of the three variants seems to preserve in all details the original formulation. The amount of polishing in the variant transmitted by the *Rasā'il ikhwān al-ṣafā'* could even permit the assumption that their authors have retransformed the language of bricks into a language of squares and tetragons. The analysis of the proofs in MS Mumbai, however, shows unequivocally that the language of bricks was superimposed over an Arabic text, which spoke of squares and rectangles.

<sup>12</sup> ومن خواصية المثلث المنفرج الزاوية ان مربع الوتر اكثر من مربع الضلعين بمقدار مربع احد الضلعين فيما هو خارج منه الى مسقط العمود مرتين

<sup>13</sup> كل مثلث منفرج الزاوية اذا اخرج احد الضلعين المحيطين بها ايهما كان الى مسقط العمود فان تلبين وترها في نفسه اكبر من تلبين كل واحد في نفسه مجموعين بمثل الضلع المخرج منه الى مسقط العمود مرتين

<sup>14</sup> كل مثلث منفرج الزاوية اذا اخرج من زاويته المنفرجة احد الضلعين المحيطين بها اياها كان الى مسقط العمود الذي يقع عليه خارج المثلث فان تلبين وتر الزاوية المنفرجة في مثله اكثر من تلبين الضلعين المحيطين بها كل واحد في مثله مجموعين بمثل تلبين الضلع المخرج منه فيما اخرج منه الى مسقط العمود مرتين

One example illustrating this claim can be found already in the very first theorem where this language is introduced, i.e., in theorem I,46. Its enunciation uses the language of bricks, while the end of its proof and the conclusion maintain the geometrical language proper.

“[For] each triangle: the *brick* of the chord of the right angle times itself is the same as the *brick* of the two remaining sides, each one times itself” [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 12b,15].<sup>15</sup>

The end of the proof:

“And with this property we explain that the *surface* *ak* is the same as the *surface* *mh*. Hence it has been made clear that the *square* *bh*, which is the *brick* *bg* times itself, is the same as the two *squares* *ha*, *ak*, which are the *brick[s]* *ab*, *ag*, each one times itself” [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 13a,2-4].<sup>16</sup>

More examples can be found almost at every step. In the definitions of Book II, the *bricks* are part of the commentary, but not of the definitions themselves. In the theorems, they appear in the proofs as appositions to *surfaces* and *squares*. The claims raised in MS Mumbai’s introduction to Book II and its concluding paragraph point unmistakably in the same direction, i.e., that the language of bricks was the work of an editor.

#### 2.4. The author of the introduction and concluding paragraph to Book II and the concept of bricks

It seems to be even possible to go a step further and argue that the author of the introduction to Book II and its concluding paragraph was also the person who introduced the concept of *talbīn* (brick) into the *Elements*.

A first argument in favor of this claim is the observation of the previous subsection, i.e., that the language of bricks is a secondary linguistic layer in the version of MS Mumbai. If this language were used by al-Ḥajjāj in his translation as an available mathematical terminology of pre-Islamic practices, the geometrical language in any later edition depending upon this translation should possess the properties of a secondary level of change. Such a relationship is, however, not supported by the textual structures characterizing the version in MS Mumbai.

A second argument in favor of my claim is the content of the introduction to Book II in MS Mumbai and its concluding paragraph. These two pieces explain the introduction of the language of bricks as the result

كل مثلث قائم الزاوية فان تلبيين وتر الزاوية القائمة في [نفسه] مثل تلبيين الضلعين<sup>15</sup>  
الباقيين كل واحد في نفسه

<sup>16</sup> وبهذه الصفة نبين ان سطح اك مثل سطح مه فقد استبان ان مربع به وهو تلبيين ج ب  
في نفسه مثل مربعي اح ك ا وهما تلبيين اب اج في نفسه

of a philosophical point of view. They state that in the order of the sciences the highest ranking one is the one, which deals with *rational objects*, i.e., objects that possess neither matter nor position. The middle science treats objects possessing no matter, but position. The lowliest science is the one that uses objects possessing both matter and position. Proklos' and Simplicios' commentaries on the *Elements* spell out, which of the two disciplines is the highest and which is the middle in MS Mumbai's classification. Unity is the object par excellence with no matter and no position. The point is the object par excellence that possesses position, but is free of matter. Hence, the highest science is number theory and the middle science is geometry. Aristotle's epistemology finally clarifies that the lowest science is physics. The entire scheme may, however, have reached Arabic-writing authors through other sources such as Neoplatonic commentaries on the *Elements*, Neopythagorean number theory, Ptolemaic astronomy, or Galenic medicine.

The introduction to Book II in MS Mumbai and its concluding paragraph emphasize further that the geometrical proofs of the theorems II,1-11 need to be replaced by *rational* proofs. Their author claims that he himself was responsible for the alternative rational proofs. He legitimizes his interference by the view that some of the objects of the middle science are close to rational objects, because the proofs for certain theorems working with such objects can be carried out without the help of forms and figures. Additionally he assures his reader that he did not embark on innovation, but rather on restoring previous knowledge of higher rank arguing that Euclid omitted the *rational* proofs, because students could not grasp their meaning. The alternative proofs in Book II deal exclusively with bricks, while the language of bricks in the original proofs extant in MS Mumbai shows clear signs of interpolation. The comments on the definitions in Book II explain Euclidean concepts by bricks and bricks in analogy by numbers. The author's aim obviously was not to introduce numbers as the *rational* objects of Book II, but the hybrid bricks, which link geometry to number theory. Bricks continue to be objects of geometry, but can be talked about in terms of arithmetic. This arithmetical aspect of bricks enables their proofs to be carried out without forms and figures. Based on these aspects of the language of bricks in Book II of MS Mumbai, the previous assumption that this language reflects a terminology from pre-Islamic mathematical practice can no longer be upheld. It seems to be more plausible that the terminology was borrowed from Nikomachos' *Introduction to Arithmetic*, i.e., from pre-Islamic philosophy of mathematics.

The philosophical stance taken in Book II in MS Mumbai opens a new page in the history of the relationship between arithmetic and geometry among mathematicians and philosophers in Islamic societies. While scholars such as Thābit b. Qurra favored geometry over arithmetic and algebra, the unknown author of MS Mumbai's version of Book II in contrast privileged number theory over geometry. Hence, the purely arithmetical versions of Book II available in various Arabic and Arab-Latin editions, translations, and fragments may indicate that the unknown author of MS Mumbai's Book II was not the only scholar who subscribed to the idea that objects without matter and position were nobler than objects without matter, but with position and opted for an arithmetical reformulation of Book II.

### ***2.5. When did the editors of MS Mumbai's version of the Elements presumably live?***

As I have argued so far, the now extant version of MS Mumbai, cleared from its glosses and marginal notes, was probably not the work of one editor alone. It is more likely to have taken its present form in a long historical process. The oldest addition was presumably the second introduction to Book I. The sheer literalness of its technical terminology in comparison with Greek descriptions of the same ideas points to a Greek origin of its composition. Its addition to the *Elements* probably also occurred in Antiquity. The youngest layer, except for the glosses and marginal notes, is the first introduction to Book I and the concluding remarks of this book, which cannot have been composed before Books XIV and XV were translated into Arabic by Qusṭā b. Lūqā. The time when the comments on the definitions, postulates, and axioms were added to Book I is difficult to ascertain. They seem to have infiltrated the text at an early moment, since they discuss the naming of the axioms for the particular form used in the version of MS Mumbai, i.e., *'ilm 'āmm jāmi'*. Attested too by Ibn Sīnā and by a gloss in the Leiden manuscript of al-Nayrīzī's commentary, the expression is thought to have been used by al-Ḥajjāj in translating the Greek *κοινὰ ἔννοιαι*. This argument for an early inclusion of the comments on the definitions, postulates, and axioms finds support in Book I's link to Ibn al-Muqaffa's work on Aristotelian logic mentioned previously.

In the case of the introduction to Book II, the use of *ṭma* for matter also points to an early period in the history of Arabic philosophical terminology. *ṭma* originally means a piece of earth and is derived from *ṭn* (earth, clay). Endress noted in his entry in the *Encyclopaedia of Islam* that as “a technical term of philosophy, *ṭma* was used in some early Arabic translations from the Greek and in the first period of Arabic philosophical writing to render the basic meaning of the Greek *ὑλη*” [Endress 2000,

p. 530]. According to Ibn Sīnā's testimony, until the late tenth century, four terms were used in Arabic to render the philosophical concept of matter, albeit in different contexts and with different shades of meaning and relevance: *ṭīna*, *jawhar*, *hayūla*, and *mādda* [Endress 2000, p. 530]. Hence, the usage of *ṭīna* as a central term for matter in MS Mumbai points to an early origin of the addition to Book II and its change to the language of bricks.

Taking all these observations together, the result is that the version of MS Mumbai may have been created by at least four editors – one for introduction 2 in Book I, one (or perhaps two) for the comments on the definitions, postulates, and axioms in Books I and II as well as for the introduction and concluding paragraph to Book II, one for the first introduction and the concluding remarks to Book I and one (or more) for the glosses in all nine books, in particular the longer quotations from Thābit b. Qurra's *Iṣlāḥ* and from al-Nayrīzī's commentary. The first editor probably lived in Antiquity. All remaining editors almost certainly lived after Ibn al-Muqaffa'. The second editor probably was a (younger) contemporary of al-Ḥajjāj. The third editor lived after 870, i.e., after Iṣḥāq b. Ḥunayn translated the *Elements* anew and after Qusṭā b. Lūqā translated Books XIV and XV. The fourth editor was either a contemporary of al-Nayrīzī or lived later in the tenth century.

According to Ibn al-Nadīm, there was – beside al-Jawharī and al-Kindī – a third scholar in the early ninth century, who wrote an edition of the *Elements*: Sanad b. 'Alī. No descriptions of his edition of the *Elements* are known. No extracts of his *Iṣlāḥ* are extant. We know that he studied with his father and later with al-Jawharī, that he was interested in astronomy, the *Almagest*, and mathematics; but we do not know whether he was at all interested in philosophy. Thus, it has to remain an open question whether he may have been the first Arab author who started the editorial process, which finally led to the version extant today in MS Mumbai. No proposals can be made with regard to any likely candidate for the third and possible fourth editor.

#### 2.6. *Does MS Mumbai represent the second version produced by al-Ḥajjāj?*

Having shown that the version in MS Mumbai can only be understood as the product of several editors, the question to be discussed now concerns the Euclidean text proper, i.e., the text of MS Mumbai without its additional material such as the introductions, glosses, and comments on definitions, postulates, and axioms. In order to answer this question, I will discuss first two issues of structure, namely the existence of porisms and the sequences of the theorems in certain books. Secondly, I will analyze theorems I,5 and 6, for which we possess three variants deviating

substantially from that transmitted in all but one extant manuscript of the primary Arabic transmission of the *Elements*: MS Mumbai, al-Nayrīzī's commentary, and MS Rabāṭ 1101, a manuscript of the Arabic primary transmission.

### 2.6.1. *Issues of structure*

Versions connected with the Ḥajjāj tradition are known for transmitting a lesser number of porisms than the Ishāq-Thābit tradition. This difference can help to evaluate the version in MS Mumbai. The incompletely preserved translation by Adelard of Bath, which differs structurally clearly from texts of the Ishāq-Thābit tradition and hence is derived in all likelihood from the Ḥajjāj tradition, contains in the first eight books (Book IX is missing) nine porisms [Busard 1983, pp. 44-45 (I,15), 75 (II,4), 89 (III,1), 103 (III,15 = Heiberg III,16), 143 (IV,15), 173 (VI,8), 180 (VI,17 = Heiberg VI,19), 199 (VII,2), 227 (VIII,2)]. Several of these porisms contain only a fraction of Heiberg's Greek form, most often the first two sentences.<sup>17</sup> MS Mumbai has six porisms [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 20b, in the margin to line 13 (III,15 = Heiberg III,16); 34b, 24-26 (IV,15); 49a,4-9 (VI,8); 51a,27-51b,2 (VI,17 = Heiberg VI,19); 56a,9-12 (VII,2); 67a,22-24 (VIII,2)]. Only one of the three missing porisms, i.e., I,15, can be said with certainty not to have been part of the version transmitted in MS Mumbai, since the folios, where the remaining two, i.e., II,4 and III,1, may have been found, are lost [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 6b,8-10; 14b,29-15a,1; 17b,29-18a,1]. Hence, a purified text of MS Mumbai can be regarded as an indirect witness of the Ḥajjāj tradition. The agreement between Adelard of Bath's translation and MS Mumbai in regard to the porisms even suggests considering this particular component as a direct witness of the structural features of al-Ḥajjāj's work.

<sup>17</sup> Gerard of Cremona's translation, linked to the Ishāq-Thābit tradition, gives thirteen porisms [Busard 1983a, cc. 13 (I,15\*), 43 (II,4), 58 (III,1), 72 (III,15 = Heiberg III,16), 115 (IV,15), 144-145 (VI,8 porism 1 and porism 2), 151 (VI,18 = Heiberg VI, 19), 152 (VI,20, porism 2, an elaborated variant of the last part, 168 (VII,2), 191 (VIII,2), 201 (VIII,14), 202 (VIII,15)]. The porism to I,15 is integrated into the text of the theorem. As in the case of Adelard of Bath's version, several of the porisms contain only a part of Heiberg's Greek forms. Gerard's second porism to VI,8 is not found in Heiberg's text. The porisms given by Gerard to theorems VIII,14 and 15 are not found in Heiberg's text, but are given as theorems. Heiberg's Greek text includes fifteen porisms, five of which are marked as interpolations [*Euclidis Elementa* 1969, pp. 24 (I,15\*), 73 (II,4\*), 95 (III,1), 119 (III,16), 138 (III,31\*), 159 (IV,5\*), 177 (IV,15); *Euclidis Elementa* 1970, pp. 12 (V,7), 29 (V,19), 56-57 (VI,8), 72 (VI,19), 76-77 (VI,20, porism 1 + porism 2\*), 108 (VII,2), 152 (VIII,2)]. Porisms marked with an asterisk are those, which Heiberg considered as interpolations.

This argument is supported by other aspects of the structural parallelism between the version of MS Mumbai and Adelard of Bath's Latin translation. Two examples will suffice to illustrate this point. In Adelard of Bath's translation, the theorems after VI,8 have the following order when numbered in the sequence of Heiberg's Greek text: VI,13, 11, 9, 10, 14-17, 19, 20, 18, 21, 22, 24 26, 23, 25, 27-30, 32, 31, 33 [Busard 1983, pp. 173–195]. The version in MS Mumbai agrees with this order exactly until it breaks off in the enunciation of VI,31 [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 49a, 11-55b,29].<sup>18</sup> In Book VIII, Adelard of Bath's translation presents the following order after theorem VIII,10: VIII,11.2+12.2, 13-16, 18, 20, 19, 21-27 [Busard 1983, pp. 235–247]. MS Mumbai follows exactly the same sequence [MS Mumbai, Mullā Fīrūz, R.I.6, ff. 70b,13-74b,29].<sup>19</sup>

The close relationship between MS Mumbai and Adelard's translation, despite all of its profound differences in their technical language and the additions found in MS Mumbai, is underlined by the fact that both omit Heiberg's theorem VIII,17 the enunciation of which they add as a follow-up at the end of Heiberg's VIII,16 simply stating that its proof is similar to the previous one [Busard 1983, 240; Euclid 1970, p. 174; MS Mumbai, Mullā Fīrūz, R.I.6, f. 72a,20-23]. Gerard of Cremona's translation also has this particular feature, but in addition gives the enunciations of Heiberg's VIII,16 and 17 as porisms to Heiberg's VIII,14 and 15 [Busard 1983a, cc. 201–202]. This observation confirms once more that MS Mumbai and Adelard's translation are derived from a shared ancestor and hence are undoubtedly witnesses of the Ḥajjāj tradition.

#### 2.6.2. *Theorems I,5 and 6*

Could the text of MS Mumbai's theorems without the porisms be regarded as a direct witness of the Ḥajjāj tradition? This question is much more difficult to answer, because we do not possess a single reliable text of this tradition, neither for al-Ḥajjāj's translation nor for his edition. Hence, it is extraordinarily difficult to make any safe statement with respect to the tradition as a whole and even more so with respect to its two different components. The following comparative analysis of theorems I,5

<sup>18</sup> Gerard's translation has the following order: VI, 13, 11, 12, 9, 10, 14-17, 19, 20, 18, 21-24, 26, 25, 27-30, 32, 31, 33 [Busard 1983a, p. XVIII].

<sup>19</sup> The structure in Gerard's translation agrees until VIII,14 with that given in Heiberg's text. The porisms to VII,14 and 15 coincide with the enunciations of VIII,16 and 17 in Heiberg's text. Heiberg's theorem VIII,16 follows in Gerard's translation at its proper place, but Heiberg's theorem VIII,17 is missing. Then Gerard's translation follows again the sequence in Heiberg's text until VIII,25 adding after it two theorems not found in Heiberg's text before it returns to the sequence found in Heiberg's text [Busard 1983a, p. XIX].

and 6 tries to unravel some of the threads tying MS Mumbai, al-Nayrīzī's commentary, and al-Ḥajjāj's work together.

In an earlier study of the then available texts, I observed that al-Nayrīzī gave two alternative formulations of the enunciation of theorem I,6 in his commentary, while the enunciation in his main text consisted of a combination of elements from these two versions [Brentjes 2001, pp. 26–27].<sup>20</sup> I concluded that al-Nayrīzī was the author of this merged form of the enunciation [Brentjes 2001, p. 27]. With the newly found version in MS Mumbai, it appears that the situation is more difficult. MS Mumbai namely has the enunciation of al-Nayrīzī's main text as its enunciation of theorem I,6. One possibility to interpret this finding is to assume that this enunciation is an interpolation in MS Mumbai, which came from al-Nayrīzī's main text. The other possibility is to assume that MS Mumbai and al-Nayrīzī's main text share a common ancestor. The proofs in MS Mumbai and al-Nayrīzī's main text differ considerably. While this situation does not exclude the possibility that MS Mumbai's enunciation has indeed come from al-Nayrīzī's main text, it also does not offer any support for it.

To deepen the discussion, we need to turn to MS Rabāṭ 1101, the one manuscript of the Arabic primary transmission that differs in its enunciation of theorem I,6 from the form transmitted by all other extant known manuscripts of this tradition as well as from all the variants found in MS Mumbai and al-Nayrīzī's commentary. The standard version of the enunciation in the other manuscripts of the Arabic primary transmission is version 2 given by al-Nayrīzī in his commentary.

MS Rabāṭ 1101:

<sup>20</sup> Version 1 of al-Nayrīzī's commentary: “[For] each triangle: (if) the two angles, which are above the base, are equal to one another, then it is isosceles.”

كل مثلث تكون الزاويتان اللتان فوق القاعدة منه متساوي الساقين

[Codex Leidensis 399.1, I.1, p. 60]; version 2 of al-Nayrīzī's commentary: “If two angles of a triangle are equal, then the two sides, which subtend the two, are equal to one another.”

إذا تساوت زاويتان من مثلث فإن الضلعين اللذين يوترانهما متساويان...

[Codex Leidensis 399.1, I.1, p. 60]; al-Nayrīzī's main text: “If two angles in a triangle are equal, then it is isosceles.”

إذا تساوت زاويتان من مثلث فهو متساوي الساقين

[Codex Leidensis 399.1, I.1, p. 60].

“If two angles in a triangle are equal to one another, then the two sides, which subtend those two angles, are equal to one another” [MS Rabāṭ, al-Maktaba al-Malikiyya, al-Khizāna al-Ḥasaniyya 1101, f. 4a, 8–9].<sup>21</sup>

MS Rabāṭ 1101's enunciation follows much more closely Heiberg's Greek text than the formulations of MS Mumbai, al-Nayrīzī's main text, and al-Nayrīzī's first version [Euclid 1969, p. 13]. As a result, its style is more elaborate than that of MS Mumbai and al-Nayrīzī's first version. Since the second part of these two versions is clearly a reformulation of the Greek phrasing of the statement as given by Heiberg, it is plausible to assume that neither version can represent directly al-Ḥajjāj's edition. On the other hand, the similarities between MS Rabāṭ 1101's version and that of the other extant manuscripts of the Arabic primary transmission as reflected in al-Nayrīzī's version 2 allows us to consider both versions as the result of a translation rather than an edition. Interior linguistic features of the first two books as transmitted in most of the manuscripts of the Arabic primary transmission support the view that MS Rabāṭ 1101, I,5–13 is not part of the Iṣḥāq-Thābit tradition. Hence, it is most plausible to assume that it is a remnant of the Ḥajjāj tradition, probably of al-Ḥajjāj's translation. If this can be supported in the future by further evidence, the enunciation of I,6 in MS Mumbai and in al-Nayrīzī's main text cannot be regarded as a direct witness of the al-Ḥajjāj's translation. If further evidence would verify that the first version of this enunciation in al-Nayrīzī's commentary comes indeed from al-Ḥajjāj's edition as I am inclined to believe, then the enunciation of I,6 in MS Mumbai and in al-Nayrīzī's main text cannot be regarded as a direct witness of the Ḥajjāj tradition, but merely as a derivative from this tradition.

As for I,5, the comparison between MS Mumbai and MS Rabāṭ 1101 shows that the former again has a simpler formulation than the latter.

MS Mumbai:

“[For] each isosceles triangle: if the two angles, which are upon its base, are equal to one another and its two equal sides are extended, then the two angles, which fall below the base, are also equal to one another” [MS Mumbai, Mullā Fīrūz, R.I.6, f. 4b,15–17].<sup>22</sup>

MS Rabāṭ 1101:

“[For] each isosceles triangle: if its two angles, which fall upon the base, are equal to one another and if from those two angles two other lines, connected

<sup>21</sup> إذا كان في مثلث زاويتان متساويتان فإن الضلعين اللذين يوتران تينك الزاويتين متساويتان

<sup>22</sup> كل مثلث متساوي الساقين فإن الزاويتين اللتين على قاعدته متساويتان وإن أخرج ضلعاها المتساويان فإن الزاويتين اللتين تقعان تحت القاعدة أيضا متساويتان

with the two equal lines, are extended in straight extension, then the two angles, which fall below the base, are also equal to one another" [MS Rabāṭ, al-Maktaba al-Malikiyya, al-Khizāna al-Ḥasaniyya 1101, f. 3b,17–19].<sup>23</sup>

Comparing these two formulations with Heiberg's Greek text shows that MS Mumbai's shorter version agrees with the Greek text except for one substantial alteration – the replacement of the Greek concept of straight lines by the Arabic concepts of sides, while MS Rabāṭ 1101 retains the original Greek concept, but adds an explanation to them [Euclid 1969, p. 12]. On the other hand, MS Mumbai and MS Rabāṭ 1101 are linked by their transforming three Greek plural forms for triangle, angle, and straight line either into a singular (triangle) or a dual (angle, straight line/side). Hence, both versions come from one and the same interpretational approach to the Greek style. Probably, MS Rabāṭ 1101 either reflects another Greek version of I,5 than MS Mumbai or represents a case, where al-Ḥajjāj filled in what he considered missing in the Greek form of the theorem. MS Mumbai's almost complete agreement with the form of I,5 in the main text of al-Nayrīzī's commentary suggests considering the first possibility as more plausible.

Al-Nayrīzī:

"[For] each triangle: if its two angles, which fall over its base, are equal to one another and if its two equal sides are extended, then the two angles, which fall below the base, are also equal to one another" [Codex Leidensis 399.1, 1893, p. 54].<sup>24</sup>

This observation implies that MS Rabāṭ 1101 may possess close links to al-Ḥajjāj's original translation, while MS Mumbai's and al-Nayrīzī's versions represent or are derived from al-Ḥajjāj's edition, an enterprise for which the translator apparently also used alternative Greek material, if my understanding of the textual features of I,5 in the three discussed sources is correct. MS Rabāṭ 1101 cannot, however, represent any stage whatsoever of the Ishāq-Thābit tradition as documented by the form of I,5 in the other manuscripts of the primary transmission.

MS Tehran, Malik 3586:

"The two angles, which are over the base of the isosceles triangles, are equal to one another and if the equal straight lines are extended, then the angles,

<sup>23</sup> كل مثلث متساوي الساقين فان زاويتيہ اللتين تقعان على القاعدة متساويتان وان اخرج من تينك الزاويتين خطان متصلان بالخطين المتساويين على استقامة فان الزاويتين اللتين تقعان تحت القاعدة ايضا متساوية

<sup>24</sup> كل مثلث متساوي الساقين فان زاويتيہ اللتين تقعان فوق قاعدته متساويتان وان اخرج ضلعاه المتساويان فان الزاويتين اللتين تقعان تحت القاعدة ايضا متساويتان

which are below the base, are equal to one another" [MS Tehran, Malik 3586, f. 252b, 13-15].<sup>25</sup>

In this Arabic rendition of I,5, the Greek plural of the triangle and straight line is preserved. The only change with regard to Heiberg's Greek text that occurred here is the switch from the Greek plural for the angle to the dual at the beginning of the enunciation, while at the end of the enunciation this plural is still in place. Hence, the approach to the Greek style chosen by Ishāq b. Hunayn and only slightly modified here either by Thābit b. Qurra or by the vicissitudes of continuous copying differs significantly from the approach manifested in MS Mumbai, MS Rabāṭ 1101, and al-Nayrīzī's main text.

An analysis of the proof of I,5 confirms what has been deduced on the basis of the enunciation alone: MS Rabāṭ 1101 undoubtedly transmits a version close to Heiberg's Greek text, while MS Mumbai's version is clearly an edited abbreviation of a text similar or even identical to the one extant in MS Rabāṭ 1101. The behavior of MS Rabāṭ 1101 in the proof of I,5 resembles its behavior in the enunciation. It occasionally adds explanatory remarks to Heiberg's Greek text. Whether these remarks are of Greek origin cannot be decided with certainty. The comparison also shows that the technical terminology used in MS Rabāṭ 1101 does not always correspond literally to the one in the Greek text. In contrast, MS Tehran Malik 3586 almost always follows literally the Greek terminology and often also the Greek style. The overall picture presented by MS Rabāṭ 1101 is, nonetheless, one of close agreement with Heiberg's Greek text. Thus, it may well be that the theorems I,5-13 in MS Rabāṭ 1101 retain major traces of al-Ḥajjāj's translation. If that were the case, these fragments indicate clearly that al-Ḥajjāj and Ishāq b. Hunayn did not follow the styles of translation, which are commonly ascribed to them, i.e., al-Ḥajjāj did not translate *verbum-ad-verbum* and Ishāq did not focus on a content oriented rendering.

As for MS Mumbai's relationship to the Ḥajjāj tradition, the analysis of theorem I,5 suggests considering the manuscript's text of its theorems and proofs without the interpolations as indeed a member of this tradition and most likely closely related to Ḥajjāj's edition. A more extended analysis of the individual theorems of Books I-IX in MS Mumbai will certainly yield more arguments for weighing the possibilities more carefully.

<sup>25</sup> الزاوتان اللتان فوق القاعدة من المثلثات المتساوية الساقين وان اخرجت الخطوط المستقيمة المتساوية فان الزوايا التي تحت القاعدة تكون متساوية

### 3. TRACES OF AN OLDER GREEK STRATUM OF THE *ELEMENTS* IN MS MUMBAI

MS Mumbai contains – as most other indirect witnesses of the Ḥajjāj tradition – several well-known traces of a more simple mathematical level than the one found in the text established by Heiberg and often confirmed by the Iṣḥāq-Thābit tradition. Examples are the omission of I,45, the form of II,14, or the proofs of theorems in Books III and IV for particular cases rather than the general case. MS Mumbai adds to this set new, previously unknown examples. They are found in Books V (theorem 22) and VI (theorems 28 and 29). In theorem V,22 Heiberg's Greek text talks of an arbitrary number of quantities having pairwise the same ratio, while MS Mumbai limits the statement to two sets of three quantities having pairwise the same ratio [Euclid 1970, p. 32; MS Mumbai Mullā Fīrūz, R.I.6, f. 44a,5-6]. In theorems VI,28 and 29, Heiberg's Greek text speaks of the construction of a parallelogram on a straight line under certain conditions, one of them being that the parallelogram be equal to a given rectilinear figure [*Euclidis Elementa* 1970, pp. 90, 93]. MS Mumbai instructs how to carry out the said construction for a parallelogram equal to a given triangle [MS Mumbai, Mullā Fīrūz, R.I.6, f. 54b,3-5 and 26–28]. In VI,28 MS Mumbai omits furthermore the condition placed upon the given rectilinear figure [MS Mumbai, Mullā Fīrūz, R.I.6, f. 54b,3-5].

The content and form of these simpler theorems agree perfectly well with the other, already known cases of simpler forms ascribed in Arabic sources to the Ḥajjāj tradition and its Greek ancestor(s). In the case of V,22, MS Mumbai's form fits the proof of the theorem as well as the form of the surrounding theorems of similar content much better than the generalized form of Heiberg's text. Hence, the structural difference found in fragments of the Ḥajjāj tradition in comparison to Heiberg's Greek text as well as to texts of the Iṣḥāq-Thābit tradition should be accepted as features of the Greek ancestor(s) of the Ḥajjāj tradition too. It may well be possible that even the two additional definitions in MS Mumbai's Book V belong to the same older Greek stratum as all the other features discussed here. As a consequence, MS Mumbai's text of the theorems and proofs cleared from interpolations is highly likely a witness to the Ḥajjāj tradition. Its properties firmly suggest that the Greek text of the *Elements* as established by Heiberg should be revised following the testimony of the available fragments of the Ḥajjāj tradition.

### CONCLUSIONS

The features of MS Mumbai as discussed in this paper and their comparison with Greek, Arabic, and Arab-Latin versions of the *Elements* leave no doubt that MS Mumbai represents the Ḥajjāj tradition. Its core text when freed from all additions and interpolations is the closest indirect Arabic witness of al-Ḥajjāj's second version that is known today. The additions found in the manuscript are mostly coming from Arabic editors of the ninth and tenth centuries. Only one of the additions came into the Euclidean text already during Antiquity. The additions to Book II document a philosophical interest in mathematics among editors of the Arabic *Elements* and indicate that an apparently practical terminology probably originated rather in this philosophical context. The core text in MS Mumbai underlines that the Greek version of the *Elements* translated and edited by al-Ḥajjāj was simpler than the Greek version upon which the Iṣḥāq-Thābit tradition relied. This insight suggests to revisit at least certain parts of Heiberg's edited Greek text on the basis of the Arabic and Arab-Latin transmission.

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