Homotopy type theory

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To Vladimir Voevodsky,
whose Univalence Axiom has opened many mathematical doors

We give a glimpse of an emerging field at the intersection of homotopy theory, logic, and theoretical computer science: *homotopy type theory*. One key ingredient of this approach is Vladimir Voevodsky's *Univalence Axiom*. It is the goal of this paper to provide a short introduction to some of the ideas of homotopy type theory and univalence. The approach taken here is to first develop some of the historical and mathematical context in which homotopy type theory arose and then to describe the Univalence Axiom and related technical machinery.

1. Introduction

The past decade has seen the birth of a new research field at the intersection of pure mathematics and theoretical computer science: *homotopy type theory*. It is in the framework of homotopy type theory, that Vladimir Voevodsky formulated his celebrated *Univalence Axiom* and stated his ideas regarding *univalent foundations*.

Homotopy type theory [4, 21, 12] is an emerging field which brings together insights from research in pure mathematics (higher-dimensional category theory, homotopy theory, mathematical logic, etc.) and computer science (type theory, programming languages, proof assistants, etc.). Whereas in the usual approach to the foundations of mathematics one takes sets as the basic entities from which mathematical structures are constructed, in homotopy type theory the basic entities are *spaces* (homotopy types), in the sense of homotopy theory, rather than sets. The key insight which led to the development of homotopy type theory is the realization that a direct axiomatization of spaces already existed in Martin-Löf's work on type theory. Because type theory, which plays an important role in mathematical logic, is the theoretical basis for many current programming languages and computer proof assistants (*Coq*, *Agda*, etc.), proofs in homotopy theory and other parts of pure mathematics can be formally verified in proof assistants using the ideas of homotopy type theory.

This approach is justified by the *homotopy theoretic interpretation* of type theory, which was independently discovered by Awodey and Warren [5, 30] and Voevodsky [27, 28], building on ideas of Hofmann and Streicher [11, 24], Moerdijk, Palmgren, and others. This interpretation relates structures arising

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in homotopy theory, such as Kan complexes and Quillen model categories, with Martin-Löf's [19, 17] dependent type theory. At around the same time, Gambino and Garner [7], van den Berg and Garner [6], and Lumsdaine [16] showed that this interpretation could be turned around to construct homotopy theoretic structures from the syntax of type theory. Crucial to this entire endeavor was Voevodsky's discovery of the *Univalence Axiom* which captures important features of his Kan complex model of type theory. (Kan complexes, which are especially nice combinatorial models of spaces and ∞-groupoids, are discussed more in Section 2.8 below.)

The following ingredients make homotopy type theory possible:

(1) The homotopy theoretic interpretation of type theory.

(2) The Univalence Axiom.

(3) Higher-inductive types or other type theoretic encodings of homotopy theoretic data.

We have already encountered items (1) and (2) above. Item (3) is roughly the type theoretic analogue of combinatorial models of spaces such as models using CW-complexes or simplicial sets. In summary, item (1) shows that it is indeed consistent to view types as spaces. Item (2) forces types to behave more like spaces and item (3) makes it possible to describe familiar spaces and constructions on spaces (spheres, suspensions, etc.) in terms of types.

The goal of the present paper is to describe in an informal way some of the ideas which have made their way into homotopy type theory. In particular, we touch, in varying levels of detail, on all three of the ingredients mentioned above.

Summary of the paper. In Section 2 we develop some of the background material which, while not strictly necessary, should provide a clearer historical and mathematical context for homotopy type theory. In particular, we describe the development of type theory in parallel with developing some simple, but important, ideas from algebraic topology. These small "vignettes" are designed to help draw out some of the analogies between homotopy theory and type theory. In particular, both homotopy theory, especially when viewed from the perspective of higher-dimensional category theory, and type theory have made use of ideas related to annotating known mathematical structures with additional data. For example, up-to-homotopy, algebraic structures are the result of taking familiar equational theories (such as the theory of groups) and replacing the equations by "annotating" homotopies. On the other hand, Church's simple type theory can be seen as the result of annotating derivations in propositional logic.

In Section 3 we introduce the Univalence Axiom and other ideas of Voevodsky regarding the foundations of mathematics. This section is the heart of the paper.

In Section 4 we make some concluding remarks regarding connections between homotopy type theory and computer assisted proof. This is something which we do not treat in detail, but which is important since it has been one of the motivating factors for Voevodsky and others working in this area.

Other relevant literature. Although there is some overlap between the present paper and our papers [21, 4], the exposition here is focussed a bit more on the logical side of matters and we here include more detail of the historical context on the logical side. For a very brief introduction to Voevodsky's Univalence Axiom we invite the reader to consult [4]. For a detailed look at homotopy type theory and univalent foundations with a view towards the formalization of mathematics in Coq we suggest [21]. The textbook [12] develops the theory more or less from first principles. Those readers interested in the semantics of homotopy type theory should look at [3, 26, 25]. A recent overview of a wider range of connections between category theory and logic can be found in [8]. Many of the ideas described in Sections 2.2 and 2.4 below are developed more fully in [9], which is one of the canonical references for type theory and its connection with logic. Voevodsky's own description of his Coq library can be found in [29].

We hope that this article will serve to persuade many readers to learn more about the fascinating emerging subjects of homotopy type theory and Voevodsky's univalent foundations program; we believe these subjects will be of mathematical interest in the long term, and that there is great potential that their further development will have a significant impact on the way working mathematicians read, write, and ddiscover proofs.

2. A few mathematical vignettes

Although homotopy type theory is a new subject, there is a long history of connections between logic, topology and algebra. A brief survey of some of these connections should help to demystify some of the concepts of homotopy type theory and will allow us to develop some of the required background incrementally. Note though that we are not historians of mathematics and, as such, the sketch presented here is admittedly biased and incomplete.

2.1. Set theoretic antimonies and Russell's theory of types

Modern "abstract" mathematics arose at around the time of Riemann's *Habilitation* lecture in 1854 on the foundations of geometry. Subsequent mathematicians such as Cantor and Dedekind developed the kind of "naïve" theory of sets and functions needed for the then new mathematics to flourish.

In 1901, Russell discovered his well-known paradox: should we admit, for each property φ, the existence of a set $\{x \mid \varphi\}$ of all elements satisfying φ, then we are led to consider the troublesome set $R := \{x \mid x \notin x\}$. In response to this and other paradoxes arising from the naïve view of sets, Russell introduced his *type theory*. In Russell's type theory, each set is to be associated with a natural number, which should be thought of as indicating the "level" at which it has been constructed. So we would write, $x : 3$, $y : 13$, ... in such a way that the relation $a \in b$ is allowed to hold *only* for $a : n$ and $b : n + 1$.[3] The natural numbers annotating the sets are called their *types*. This theory is usually taken to be the historical origin of type theory, although some of the ideas undoubtedly go back much further and modern type theory is quite different from Russell's theory. Ultimately, Russell's type theory failed to attract much of a following and the *de facto* foundational system eventually became, and still remains, Zermelo-Frankel set theory with the Axiom of Choice. Nonetheless, it is worth mentioning Russell's theory as the idea

[3] The notation $x : 3$ is not Russell's, but is closer to notation we will employ later. Also, we are simplifying matters: Russell in fact introduced several theories of types.

of *annotating* mathematical structures is present throughout the history of type theory.

2.2. Propositional logic

Although mostly known for his work in topology, Brouwer advocated an approach to foundations starkly at odds with the set theoretic trend of the time. Brouwer himself was against the formal codification of mathematical principles, but his student Heyting formalized his ideas as what is called *intuitionistic logic*. We will briefly recall the simplest form of intuitionistic logic which is called *propositional (intuitionistic) logic*.

Assume given a set $\mathcal{V}$. We refer to elements of $\mathcal{V}$ as *propositional variables*. Usually these are thought of as representing declarative statements such as, "2+2=4", "The normal distribution is symmetric about its mean", and so forth. A *proposition* is then either a propositional variable or is obtained by applying the *logical connectives* to other propositions. Here the logical connectives are summarized in Figure 1 below, where A and B denote propositions and $\perp$ is the constant false proposition (i.e., a nullary connective). We can define the negation "not A" of

	name of connective	*read as*
$\perp$	falsum	"false"
$A \wedge B$	conjunction	"A and B"
$A \vee B$	disjunction	"A or B"
$A \to B$	conditional	"If A, then B"

FIGURE 1. Logical connectives.

a proposition A as $\neg A := A \to \perp$. The set $\mathcal{P}$ of propositions is the smallest set containing $\mathcal{V}$ and closed under the operations $\neg$, $\wedge$, $\vee$ and $\to$.[4] In what follows we will restrict our attention to the portion of propositional logic concerned only with $\wedge$ and $\to$.

We now define a set $\mathcal{C}$ of what are called *contexts*. By definition a context, written $\Gamma, \Delta, \Theta, \ldots$, is a finite list $(A_1, \ldots, A_n)$ of propositions where we allow $n = 0$ (the *empty context*) and we allow $A_i = A_j$ for $i \neq j$.[5] Given a context Γ and a proposition A we will write Γ, A for the result of appending A to the list Γ.

Next, we introduce a relation $\Gamma \vdash B$ between contexts Γ and propositions B. This relation is the subset of $\mathcal{C} \times \mathcal{P}$ determined by the *inference rules* of logic. For example, there is an inference rule, called $\wedge$-*introduction*, which says that if $\Gamma \vdash A$ and $\Gamma \vdash B$, then $\Gamma \vdash A \wedge B$. It is often inconvenient to describe inference rules in such a verbose way so we use the common shorthand for such a rule as follows:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge \text{ introduction}$$

[4] To be precise, we should also require some convention regarding the use of parentheses. However, we omit such logical arcana and refer the reader to any standard text in logic for a detailed treatment.

[5] The reader will notice that we should allow several operations (*structural rules*), such as rearranging the order of the propositions occurring in the context, on contexts. These issues are important, but we have chosen to avoid them in the interest of simplifying our presentation.

The idea being that the relation below the line holds when the relations listed above the line hold. The name of the rule is indicated on the right. The additional rules governing $\wedge$ and the rules for $\to$ are summarized in Figure 2. We also require the following rule,

$$\frac{}{\Gamma, A \vdash A}\ \text{Axiom}$$

which allows us to infer that A holds when it appears in the context. One usually

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\ \text{elimination (left)}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\ \text{elimination (right)}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to\ \text{introduction}$$

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \to\ \text{elimination}$$

FIGURE 2. Remaining rules of inference for $\wedge$ and $\to$.

works with these rules by stacking one on top of another to form a tree. For example, to prove that the relation $\big(A \to (B \wedge C)\big) \vdash \big((A \to B) \wedge (A \to C)\big)$ holds we simply construct the tree illustrated in Figure 3 where, for notational convenience, we set

$$P := A \to (B \wedge C),$$

and where the leaves are instances of the Axiom rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, A \vdash A \to (B \wedge C)} \qquad \dfrac{}{P, A \vdash A}}{P, A \vdash B \wedge C}\to\text{E}}{P, A \vdash B}\wedge\text{E}}{P \vdash A \to B}\to\text{I} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P, A \vdash A \to (B \wedge C)} \qquad \dfrac{}{P, A \vdash A}}{P, A \vdash B \wedge C}\to\text{E}}{P, A \vdash C}\wedge\text{E}}{P \vdash A \to C}\to\text{I}}{P \vdash (A \to B) \wedge (A \to C)}\wedge\text{I}$$

FIGURE 3. Derivation tree of $A \to (B \wedge C) \vdash (A \to B) \wedge (A \to C)$.

Remarkably, the system formalized by Heyting can be interpreted using the open sets of a topological space X by letting $\wedge$ be intersection, $\vee$ be union, $\bot$ be the empty set, and by defining $\to$ as follows:

$$U \to V := \bigcup\{W \mid U \cap W \subseteq V\} = \big((X - U) \cup V\big)^{\circ}$$

for open sets U and V. Consequently, it is quite easy to prove that the law of excluded middle $\vdash P \vee \neg P$, which holds in *classical* propositional logic, does not hold in intuitionistic propositional logic. E.g., taking as our ambient space $\mathbb{R}$ and letting P be the open set $(0, +\infty)$ we see that $P \vee \neg P = (0, +\infty) \cup (-\infty, 0)$.

2.3. Weakening and annotating: a topological example

Interestingly, in both topology and logic several important structures have arisen by virtue of annotating known structures with an additional mathematical layer. In some cases, this process has allowed for the original intended structure to be weakened in the sense that, e.g., equations governing the structure are now relaxed and replaced by a relation other than equality.[6] The most natural setting for this kind of machinery is higher-dimensional category theory, but we will here constrain ourselves to examples of this phenomenon coming from topology and logic.

Let X be a topological space and let $x_0 \in X$ be a point of X. The *loop space of X based at x_0*, which we write as $\Omega(X)$ when x_0 is understood, is the space of continuous maps $f: I \to X$ such that $f(0) = x_0 = f(1)$, where $I = [0,1]$ is the unit interval. We define a *concatenation $f \cdot g$ operation* on $\Omega(X)$ by letting

$$(f \cdot g)(t) := \begin{cases} f(2t) & \text{for } t \in [0, \frac{1}{2}] \\ g(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

for $f, g \in \Omega(X)$. Similarly, we have an inverse operation f^{-1} which traverses f in reverse. With the structure just described $\Omega(X)$ is *almost* a group with multiplication $(f \cdot g)$, inverses f^{-1} and unit element the constant loop $r(x_0)$ given by $r(x_0)(t) := x_0$ for $t \in I$. However, it is easy to see that this is not a group. E.g., the multiplication is not associative since $(f \cdot (g \cdot h))$ will still be traversing f at $\frac{3}{8}$, whereas $((f \cdot g) \cdot h)$ will have already moved on to g at this point. One solution to this problem is to observe that although $\Omega(X)$ is not a group, it is a group *up to homotopy*.

Recall that if f and g are continuous maps $X \to Y$ for X and Y topological spaces, a *homotopy* φ from f to g (written $\varphi: f \simeq g$) consists of a continuous map $\varphi: X \times [0,1] \to Y$ such that $\varphi(x, 0) = f(x)$ and $\varphi(x, 1) = g(x)$. Coming back to the loop space $\Omega(X)$ we see that we have homotopies

$$\alpha_{f,g,h} : ((f \cdot g) \cdot h) \simeq (f \cdot (g \cdot h)),$$
$$\iota_f : (f \cdot f^{-1}) \simeq r(x_0),$$

and so forth, for $f, g, h \in \Omega(X)$.

So, we have *weakened* the group laws and replaced the equations specifying what it is to be a group by homotopies which provide a corresponding *annotation*. We sometimes say that the homotopies are *witnesses* of the algebraic laws in question. One can weaken many algebraic structures in similar ways and this has in fact been done. From our perspective the importance of this particular example will soon become clear.

[6] The terminology of *weak* versus *strict* is used frequently in the literature on higher-dimensional category theory, but is rarely clarified. In practice, a structure being strict usually refers to the case where the defining conditions of the structure are axiomatized using the smallest possible relation or relations satisfying certain constraints. Often the constraint is simply that the relation should be an equivalence relation and strictness then refers to the case where the structure is axiomatized in terms of equations. By contrast, the weak case occurs when the structure is axiomatized using a larger relation satisfying the constraints.

2.4. Annotating: a logical example

Following the early use of the locution "type theory" by Russell, the next major development in type theory came with Church's *simple type theory*. Consider for a moment a derivation in propositional logic such as the derivation from Figure 3 above. Simple type theory can be regarded as an annotated version of the $\wedge, \rightarrow$ fragment of intuitionistic propositional logic described above, where the annotations capture the structure of the derivation trees. We will write our annotations as

$$a : A$$

where A can, for now, be thought of as a proposition and a is regarded as a kind of "code" for the derivation tree of A (assuming A is derivable). For example, we would then annotate the introduction rule for $\wedge$ as follows:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \wedge B}$$

Here we use a tuple notation (a, b) for the annotation to indicate that a derivation of $A \wedge B$ can be obtained by simply collecting a derivation of A together with a derivation of B. This notation is also reminiscent of elements of a cartesian product of sets $A \times B$ and, in fact, this is the standard notation for $A \wedge B$ in simple type theory. As such, we henceforth write $A \times B$.

In simple type theory we do not speak of propositions and annotations. Instead when faced with $a : A$ we say that A is a *type* and a is a *term of type A*. A context Γ is now a list $(x_1 : A_1, \ldots, x_n : A_n)$ where the x_i are variables. The rules of inference, aside from the introduction rule for $\times$ given above, for $\times$ and $\rightarrow$ in simple type theory are summarized in Figure 4. In the introduction rule for $\rightarrow$, the variable x in $\lambda_{x:A} b$ is bound. Intuitively, $\lambda_{x:A} b$ is an algorithm for transforming derivations of A into derivations of B using b. It is sometimes convenient to think of $A \rightarrow B$ as being like the set of all functions from a set A to a set B. In this case, $\lambda_{x:A} b$ is the function given by $x \mapsto b(x)$.

$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_1(p) : A} \times\mathrm{E}\ (\text{left}) \qquad \frac{\Gamma \vdash A \times B}{\Gamma \vdash \pi_2(p) : B} \times\mathrm{E}\ (\text{right})$$

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda_{x:A} b : A \rightarrow B} \rightarrow\mathrm{I} \qquad \frac{\Gamma \vdash f : A \rightarrow B \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B} \rightarrow\mathrm{E}$$

FIGURE 4. Rules of inference for $\times$ and $\rightarrow$ in simple type theory.

So we see that simple type theory can be viewed in (at least) two ways. In the first, we think of the terms as codes for derivation trees in propositional logic. This interpretation of simple type theory is sometimes called *propositions as types* or the *Curry-Howard correspondence*. In the second, we think of types as sets and terms as elements. There is actually a third way that is worth knowing about, even if we do not make use of it here, which is to think of types as objects in a cartesian closed category and terms as generalized elements. (Conversely, the morphisms of a cartesian closed category can be regarded as equivalence classes of proofs in a formal system and we refer the reader to [14] for a detailed exposition of these

type former	*name*	*logical*	*set theoretic*
$A \times B$	simple product	$A \wedge B$	cartesian product
$A \to B$	exponential	$A \to B$	function space B^A

FIGURE 5. Products and exponentials.

matters.) The first two of these interpretations are summarized in Figure 5. In addition to being able to construct terms of given types $a : A$, we can also reason about equality of terms and types in type theory. This is taken as two further relations in addition to the basic relation $\Gamma \vdash a : A$. Namely, we write $\Gamma \vdash A = B$ to indicate equality of types and $\Gamma \vdash a = a' : A$ equality of terms. Simple type theory then incorporates what are called *conversion* or *computation* rules such as

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \pi_1(a, b) = a : A}$$

which describe the behavior of equality (which is of course also required to be an equivalence relation). The relation $\Gamma \vdash a = a' : A$ between a and a' is sometimes referred to as *definitional* (or *intensional*) equality, as opposed to the *propositional* (or *extensional*) equality which we will now describe.

2.5. Dependent type theory

The form of type theory with which homotopy type theory is concerned is sometimes called *dependent type theory* and it is due to Martin-Löf [19, 17, 18], who built on related work by Curry, Howard, Tait, Lawvere, de Bruijn, Scott and others. From the logical point of view, dependent type theory extends the analogy between simple type theory and propositional logic to logic with quantifiers ($\forall$ and $\exists$). From the point of view of foundations and set theory, dependent type theory can be viewed as a theory in which indexed families of sets are manipulated.

Now, when we are given, say, $x : B \vdash E$ we think of this as indicating that E is B-indexed or, in more geometric language, that E is fibered over B. In order to emphasize this intuition we will often write $x : B \vdash E(x)$ which is reminiscent of the indexed set notation $(E_x)_{x \in B}$. A term $x : B \vdash s(x) : E(x)$ would then be thought of as a B-indexed family of elements $s_x \in E_x$. From a category theoretic perspective it is common to think of this as describing a map $E \to B$ (cf. [23]). Instead of just $\times$ and $\to$, we now have indexed versions of these, written as $\sum$ and $\prod$ respectively, and we have rules governing when they can be introduced into a derivation. For example:

$$\frac{\Gamma, x : B \vdash E(x)}{\Gamma \vdash \sum_{x:B} E(x)}$$

The logical and set theoretic readings of these are summarized in Figure 6. We also consider notions of equality for types and terms.

In addition to $\prod$ and $\sum$, Martin-Löf introduced what is known as the *identity type*. The rule for forming an identity type is as follows:

$$\frac{\Gamma \vdash A}{\Gamma, x : A, y : A \vdash \mathtt{Id}_A(x, y)}$$

type former	*name*	*logical*	*set theoretic*
$\prod_{x:B} E(x)$	dependent product	$(\forall x \in B)E(x)$	$\prod_{x\in B} E_x$
$\sum_{x:B} E(x)$	dependent sum	$(\exists x \in B)E(x)$	$\sum_{x\in B} E_x$

where indexed products and sums of sets are given by:

$$\prod_{x\in B} E_x := \{f\colon B \to \bigcup_{x\in B} E_x \mid f(b) \in E_b\}$$

$$\sum_{x\in B} E_x := \{(x, e) \mid x \in B \text{ and } e \in E_x\}$$

FIGURE 6. Dependent products and sums.

The identity type captures a certain notion of equality between terms and it is often suggested that one should think of a term $p : \mathtt{Id}_A(a, b)$ as corresponding to a proof that a and b are equal. When such a p exists, we sometimes say that a and b are *propositionally equal*.

In the set theoretic interpretation,

$$\mathtt{Id}_A(a, b) := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise,} \end{cases}$$

and therefore propositional equality in this case coincides with definitional equality $a = b : A$. Nonetheless, it does not follow from the rules of type theory that these two notions of equality must be the same.[7] Indeed, this interpretation unnecessarily destroys the interesting structure present in the identity type.

2.6. Groupoids in topology

Earlier we saw that to each space X and point x_0 we could associate the loop space $\Omega(X)$ and that $\Omega(X)$ is a kind of group, up to homotopy. It turns out that there is a good deal more structure lurking in this example. To see this, define a groupoid $\Pi_1(X)$ called the *fundamental groupoid of* X as follows. Recall that a groupoid is a category in which every arrow is invertible. E.g., if G is a groupoid and x is an object of G, then the set $\hom_G(x, x)$ is a group. In our case, the objects of $\Pi_1(X)$ are points of the space X. Given points x and y of X, an *arrow* from x to y is then defined to be an equivalence class $[f]$ of paths $f\colon I \to X$ with $f(0) = x$ and $f(1) = y$. Here the equivalence relation is homotopy relative to endpoints. By considerations from Section 2.3 we see that this is indeed a groupoid. Moreover, for a point x of X we recover the Poincaré fundamental group $\pi_1(X, x)$ as $\hom_{\Pi_1(X)}(x, x)$.

In fact, there is no need to quotient at this stage. Instead we can follow Grothendieck and consider the entire graded set $(X_n)_{n\geqslant 0}$ with $X_0 := X$, X_1 the

[7] There are many ways to see this. E.g., type checking in the form of type theory in which these notions of equality have been identified (i.e., the theory of [18]) is undecidable, whereas, by Martin-Löf's normalization theorem, it is decidable in the theory where they have not been identified.

paths in X, X_2 the homotopies of paths relative to endpoints, X_3 the homotopies of homotopies of paths relative to endpoints, and so forth. The resulting structure is not a groupoid, but an ∞-groupoid: *the fundamental ∞-groupoid $\Pi_\infty(X)$ of X*. Moreover, Grothendieck noted that, when suitably defined, $\Pi_\infty(X)$ should capture all of the homotopy theoretic content of X. This is something which is certainly *not* true of the usual $\Pi_1(X)$. Instead, $\Pi_1(X)$ is only able to completely capture the X which are *homotopy 1-types*.

2.7. Groupoids in logic

The birth of homotopy type theory can be traced back to ideas of Hofmann and Streicher [11] (see also [24]). Hofmann and Streicher associated to any type A in dependent type theory a groupoid, which we now denote by $\Pi_1(A)$ (Hofmann and Streicher did not use this notation), by letting the objects be terms of type A and the arrows be equivalence classes $[f]$ where

$$\vdash f : \mathtt{Id}_A(a, b)$$

and the equivalence relation $\simeq$ is given by setting $f \simeq g$ provided that $f : \mathtt{Id}_A(a, b)$ and $g : \mathtt{Id}_A(a, b)$, for some $a : A$ and $b : B$, and there exists some

$$h : \mathtt{Id}_{\mathtt{Id}_A(a,b)}(f, g).$$

Moreover, Hofmann and Streicher constructed a model of dependent type theory in the category of groupoids and observed that $\Pi_1(A)$ does not seem to completely capture the content of the type A.

2.8. Early results in homotopy type theory

In 2001, Moerdijk conjectured, during the question session at the end of a talk he delivered at the Mittag-Leffler Institute, that there might be a connection between Quillen model categories, from homotopy theory, and type theory. Later, Awodey and Warren [5, 30], and Voevodsky [27, 28] independently realized how to interpret type theory using ideas from homotopy theory (in the former case, following Moerdijk's suggestion to make use of Quillen's machinery, and in the latter case using simplicial sets). We refer to our article [21] for further discussion of many of these topics, and especially regarding groupoids and homotopy n-types.

The basic idea of the homotopy theoretic interpretation of type theory, which is one of the ingredients making homotopy type theory possible, is to regard types neither as propositions nor as sets, but as spaces. In the dependent setting this means that a dependent type $x : B \vdash E(x)$ would be interpreted as a fibration over B. A fibration is a continuous map which satisfies a certain homotopy lifting property. Identity types can then be interpreted using path spaces A^I. Interestingly, the features required to interpret type theory are sufficiently general that they can be validated in a wide range of settings where homotopy theory makes sense. In particular, Quillen introduced the notion of *model category* as an abstraction of the structure which makes the homotopy theory of spaces and simplicial sets possible. The result for type theory is then as follows:

Theorem 2.1 (Awodey and Warren [5, 30]). *In any well-behaved Quillen model category there is a model of type theory.*

This perspective serves to clarify some features of dependent type theory which had previously been mysterious. Moreover, this opens the door to being able to manipulate spaces directly, without first having to develop point-set topology or even define the real numbers. That is, homotopy type theory interprets type theory from a homotopical perspective.

Voevodsky instead investigated a special model (sometimes called the *univalent model*) which he discovered in the category of simplicial sets:

Theorem 2.2 (Voevodsky [28])**.** *Assuming the existence of Grothendieck universes (sufficiently large cardinals), there is a model of type theory in the category of simplicial sets in which types are interpreted as Kan complexes and Kan fibrations.*

Kan complexes are one model of the notion of ∞-groupoid and so this result is in line with the intuition of types as corresponding to spaces and ∞-groupoids. Indeed, from the point of view of homotopy theory, Kan complexes correspond precisely to spaces (in the technical sense that there is a Quillen equivalence between the category of spaces, with its usual model structure, and the category of simplicial sets with the Kan complex model structure). The work described above, together with theorems of Gambino and Garner, van den Berg and Garner, Lumsdaine, Moerdijk and Palmgren, Streicher, as well as further results from [27, 28] and [30] constitute the earliest results in homotopy type theory.

3. Homotopy type theory and foundations

With some of the relevant background covered, we are now in a position to describe Voevodsky's univalence axiom and some of the other ideas underlying his *univalent foundations* program.

3.1. The univalence axiom

For spaces, a map $f\colon X \to Y$ is a *weak equivalence* when it induces isomorphisms on the homotopy groups. Voevodsky realized that this notion could be captured type theoretically. In order to describe this approach let us first develop some notation and basic definitions. First, we will often omit explicit mention of contexts Γ in what follows in order to avoid notational clutter. Next, note that for types A and B which are not dependent (meaning they are not parameterized by one another) we can form the simple type $A \to B$ as $\prod_{x:A} B$ (cf. the set theoretic interpretation). Now, in homotopy type theory, $A \to B$ is just the space of continuous maps from A to B.

Definition 3.1. Given $f\colon A \to B$ and $b : B$ we denote by $f^{-1}(b)$ the type defined by

$$f^{-1}(b) := \sum_{a:A} \mathtt{Id}_B(f(a), b).$$

This is sometimes called the *homotopy fiber of f over b*.

Definition 3.2. Given types A and B, a term $f\colon A \to B$ is a *weak equivalence* when there exists a term of the type

$$\prod_{b:B} \sum_{c:f^{-1}(b)} \prod_{a:f^{-1}(b)} \mathtt{Id}_{f^{-1}(b)}(c, a). \tag{3.1}$$

It is perhaps useful to consider the logical analogue of (3.1):

$$(\forall b \in B)(\exists c \in f^{-1}(b))(\forall a \in f^{-1}(b))c = a$$

which is the same as

$$(\forall b \in B)(\exists c \in f^{-1}(b))(\forall a \in A)b = f(a) \to c = a.$$

This can be decomposed into the following observations:

(1) f *is surjective*: $(\forall b \in B)(\exists c \in A)f(c) = b$.
(2) f *is injective*: $(\forall c \in A)(\forall a \in A)f(c) = f(a) \to c = a$.

That is, being a weak equivalence is the homotopy theoretic version of being a bijective function.

Definition 3.3. Given types A and B, denote by $A \simeq B$ the *type of all weak equivalences from* A *to* B. That is,

$$A \simeq B := \sum_{f:A\to B} \prod_{b:B} \sum_{c:f^{-1}(b)} \prod_{a:f^{-1}(b)} \mathtt{Id}_{f^{-1}(b)}(c, a).$$

Now, it is common in type theory to consider a type U which behaves as a kind of universe of "small types". From the point of view of homotopy theory this corresponds to the assumption of a Grothendieck universe of "small spaces". Voevodsky observed that, by virtue of the rules governing identity types, there exists a map $\iota_{A,B}$ indicated as follows:

$$\mathtt{Id}_U(A, B) \to (A \simeq B)$$

for small types A and B. Voevodsky then proved the following:

Theorem 3.4. *(Voevodsky* [28, 13]*) In the simplicial set model of type theory, the map $\iota_{A,B}$ is a weak equivalence for all A and B.*

In the simplicial set model the identity type $\mathtt{Id}_U(A, B)$ is interpreted as the space of all paths in the space U from A to B. So this result characterizes such paths as corresponding to weak equivalences $A \to B$. Now, all of the parts of this theorem can be understood type theoretically and so Voevodsky turned this result around by adding it as an axiom to type theory:

Axiom 3.5. (Univalence Axiom, Voevodsky [28]) Given U the universal type, there exists a term υ of type

$$\mathtt{Id}_U(A, B) \simeq (A \simeq B)$$

such that the underlying term $\mathtt{Id}_U(A, B) \to (A \simeq B)$ of υ is $\iota_{A,B}$.

We now consider consequences of the Univalence Axiom.

3.2. Univalent foundations

We will now describe what Voevodsky has called *univalent foundations*. To be clear, there is no implicit foundational claim made by homotopy type theory and some of the researchers in this area are satisfied with standard set theoretic foundations using Zermelo-Frankel set theory with the axiom of choice (ZFC) (and perhaps additional principles such as Grothendieck universes). So there is a sense in which homotopy type theory is entirely agnostic regarding matters of foundations. At the same time, Voevodsky has advocated for what he calls univalent foundations. Roughly, univalent foundations is simply the idea of taking homotopy type theory as a foundation for mathematics in place of alternatives such as ZFC. We will now describe some of the ideas which make this proposal reasonable.

We say that a type A is *contractible* when there exists a term of the following type:

$$\sum_{c:A} \prod_{a:A} \mathtt{Id}_A(c, a).$$

Written logically this says $(\exists c \in A)(\forall a \in A)c = a$. I.e., it says that A is a singleton set. (Note that a map $f : A \to B$ is a weak equivalence when all of its homotopy fibers are contractible.) From the point of view of homotopy theory though it merely states that the type A can be continuously deformed to a one point space. Following Voevodsky [28], let us say that a contractible type A *has h-level* 0. In general, we say that a type A *has h-level* $(n + 1)$ when the type $\mathtt{Id}_A(a, b)$ has h-level n for any terms a and b of type A.

So, for example, A has h-level 1 when $\mathtt{Id}_A(a, b)$ is contractible for all terms a and b. This is true when A is the empty type (empty space) 0 and it is also easy to see that this is true when A is itself contractible. In fact, these are the only two possibilities. That is, the spaces of h-level 1 look like the boolean values 0 and 1 from the homotopical perspective. Moving up in dimension, we see that the A which have h-level 2 correspond to sets (discrete spaces). In general, the types of h-level $(n + 2)$ correspond to homotopy n-types.

Now we are in a position to relate these ideas to foundational matters. The crucial point is that the sets sit among all of the types: we have carved them out as the types of h-level 2. The next important point here is that there are many types with h-level greater than 2. As such, if we adopt homotopy type theory as a foundational system, then our foundational system provides us access to more general spaces than just the discrete ones *at the outset*. This is in contrast with the situation in set theoretic foundations where one must build up all of the machinery of point set topology before having access to more general spaces. The final point worth making in this connection is that we have so far made no further assumptions regarding the properties of the types of h-level 2, but it is consistent that they obey the usual properties of sets in ZFC. In particular, the sets in Voevodsky's simplicial set model of type theory are indeed the classical sets of ZFC. Should we choose to adopt this perspective we should more properly see univalent foundations as being consonant with classical set theoretic foundations.[8] (We refer the reader to [22] for more regarding the properties of sets in a general model of homotopy type theory.)

[8] It is worth remarking that the authors prefer to remain agnostic about foundational matters such as these. We are merely clarifying some of the open possibilities.

3.3. Higher-inductive types

In algebraic topology it is often convenient to know that the spaces with which one is concerned have been constructed in a certain way. Consider for example CW-complexes. A *CW-complex* is a space which is built up inductively by "gluing cells". Specifically, a space X is a CW-complex if there exists a sequence of spaces X_n, for $0 \leqslant n$, and a sequence of families of continuous maps $f_i^n \colon S^{n-1} \to X_n$ called the *attaching maps* such that $X = \bigcup_{i=0}^{\infty} X_i$, X_0 is a discrete space, and either $X_{n+1} = X_n$ or X_{n+1} is the pushout of the map $\coprod_i S^n \to X_n$ induced by the attaching maps along the map $\coprod_i S^n \to \coprod_i D^{n+1}$. Working with CW-complexes as opposed to general spaces has the advantage that they have been constructed iteratively in a simple combinatorial fashion. Indeed, it is common in algebraic topology to take *space* to simply mean *simplicial set*, where simplicial sets can be understood as *purely* combinatorial descriptions of spaces (in terms of points, intervals, triangles and their higher-dimensional generalizations).

Therefore, in order to be able to manipulate familiar spaces in homotopy type theory one approach is to find a way to give a combinatorial description of spaces in type theory. This is exactly what is accomplished by higher-inductive types. In particular, using higher-inductive types it is possible to carry out all of the familiar constructions of and on spaces that one is familiar with from algebraic topology. Although we will not give a detailed description of higher-inductive types here, the idea is very simple. The combinatorial structure required to represent spaces as CW-complexes are spheres S^n and discs D^{n+1}, and the combinatorial structure required to represent spaces as simplicial sets are points, intervals, triangles, tetrahedra, *et cetera*. To represent spaces in type theory we use instead the kinds of combinatorial structure provided by identity types: paths, homotopies between paths, and so forth. The magic behind this is that the higher-inductive types are defined in such a way that they possess, by virtue of their type theoretic specifications (in terms of inference rules), universal properties which state that they are the smallest spaces possessing the required non-trivial paths, homotopies, and higher homotopies. We refer to [12] for a more detailed introduction to higher-inductive types.

4. Conclusion

In conclusion, we would like to make several remarks regarding the connection between homotopy type theory and computer proof assistants. In connection with his work in homotopy type theory and univalent foundations, Voevodsky has advocated for the formalization of mathematics in computer proof assistants, as well as for greater interaction between the developers of proof assistants and mathematicians. Voevodsky has himself written code documenting topics ranging from the homotopy theory, to the formalization of abstract algebra (see [29] for a survey of his library). In essence, there are two good reasons for doing so. First, as more mathematicians become engaged with the developers of computer proof assistants it is inevitable that the technology will adapt to better suit the needs of mathematicians. Second, it is likely inevitable that computer proof assistants will one day be as common as such tools as TEX now are and we will reach this point sooner through greater engagement by the mathematical community. For some mathematicians, these tools will already be of use. In the case of reasoning involving homotopy theoretic or higher-dimensional algebraic structures, which sometimes have large amounts of "book-keeping" involving complex combinatorial data, being able to make use of the computer to ensure that calculation errors have not been made is potentially quite useful. It is with these and like considerations in mind that Voevodsky and others have been making use of the proof assistants already to formally verify mathematical proofs in homotopy theory, algebra and other areas. Examples include the recent Coq proof of the Feit-Thompson odd order theorem [10], Voevodsky's own Coq library, and [20].

Part of the reason that homotopy type theory is closely related to work in proof assistants is that Martin-Löf type theory provides the theoretical basis of both the Agda and Coq proof assistants. As such, it is easy to work in homotopy type theory in these systems. Examples of the kinds of results which have been formalized in this setting include [1, 2, 15] as well as further results which are mentioned in the textbook [12]. We encourage the interested reader to consult these and the other references mentioned throughout this paper in order to learn more about this exciting area.

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