MICHAEL R. HERMAN

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Over the past thirty years, the work of Michael Herman and his students has had a profound impact on the development of dynamical systems, particularly that part related to stability and randomness of motion in Hamiltonian systems.

Poincaré discovered the existence of randomness (or chaos, as it is usually called nowadays) in the Newtonian three body problem. Conversely, KAM theory proves the existence of a set of stable motions in Hamiltonian systems such as the three body problem that are small perturbations of integrable systems. The stable motions fill out a set which is « large » in the sense of measure, whereas it seems likely that the chaotic motions fill out a set which is « large » in the sense of topology. Thus, one has stable and chaotic motions side-by-side. This complicated situation gives rise to many interesting problems, both numerical and theoretical. In recent decades, there has been a great deal of activity in this area.

Herman has been a leader of these developments on the theoretical side. At the Berlin International Congress of Mathematicians in 1998, he gave a talk, « Some Open Problems in Dynamical Systems », which gave an overview of his interests : Siegel singular disks, invariant tori, measure preserving diffeomorphisms of 2-manifolds, entropy and exponents, existence of periodic orbits, and instabilities of Hamiltonian flows and the problem of topological stability. These are subjects to which he or his students have made major contributions. The common theme of these subjects is stability or randomness of motions.

In his thesis, Herman solved a problem suggested by KAM theory. This problem had been posed by V.I. Arnold more than a decade earlier. It concerned an orientation preserving diffeomorphism f of the circle $\mathbb{R}/\mathbb{Z}$. Such a diffeomorphism may be lifted to a diffeomorphism $\tilde{f}$ of $\mathbb{R}$ such that $\tilde{f}(x+1) = \tilde{f}(x) + 1$. The Poincaré rotation number $\rho(\tilde{f})$ is defined to be $\lim_{n\to\pm\infty} \tilde{f}^n(x)/n$. It is easy to see that this limit exists and is independent of x. In addition, $\rho(f)$ is defined to be $\rho(\tilde{f})$ (mod.1). This is an invariant of topological conjugacy : if h is an orientation preserving homeomorphism of $\mathbb{R}/\mathbb{Z}$, then $\rho(hfh^{-1}) = \rho(f)$. Poincaré showed that f has periodic points if and only if $\rho(f)$ is rational. If the set of periodic points is not empty and not the whole circle, then f is obviously not conjugate to a rotation. This provides examples of analytic diffeomorphisms which are not topologically conjugate to a rotation.

However, Denjoy proved that if f is C^2 and $\rho(f)$ is irrational, then f is topologically conjugate to a rotation.

There is not much choice for the conjugating homeomorphism. Let $R_\alpha(x) = x + \alpha$, for $x, \alpha \in \mathbb{R}/\mathbb{Z}$. It is easy to see that if h and h_1 are conjugating homeomorphisms, then $h_1 = R_\alpha h$, for some $\alpha \in \mathbb{R}/\mathbb{Z}$. Thus it is meaningful to speak of analyticity or differentiability of the conjugating homeomorphism, without specifying which homeomorphism.

The first result concerning regularity was obtained by Arnold, who applied the fast iteration method of Kolmogorov. In the works of these authors, Diophantine conditions play an important role. A number α is said to satisfy a *Diophantine condition* if there exist $\nu, C > 0$ such that $|q\alpha - p| \geqslant Cq^{-\nu}$ for all numbers p, q with $q \geqslant 1$. The number ν is called the *exponent* and C the *coefficient*.

Arnold showed that if $\rho(f)$ satisfies a Diophantine condition and f is an analytic diffeomorphism close to a rotation in the C^ω topology, then h is analytic. Later Rüssmann showed that it is possible to prove Arnold's theorem with an arithmetic condition on $\rho(f)$ which is weaker than the Diophantine condition. However, in his original article, Arnold showed that some condition on $\rho(f)$ other than irrationality is needed.

Arnold posed the question : is it possible to remove the assumption that f is close to a rotation, under a suitable condition on $\rho(f)$?

Herman showed that it is, under the hypothesis that $\rho(f)$ satisfies a Diophantine condition with exponent 1. At the same time, he proved a differentiable version of his theorem : if f is C^r, $r \geqslant 4$, and $\rho(f)$ satisfies a Diophantine condition with exponent 1, then h is C^{r-3}. In this way, he generalized Moser's differentiable version of Arnold's theorem, at the same time that he generalized Arnold's theorem.

It does not appear to be possible to prove Herman's global theorem by the fast iteration methods that Arnold and Moser used to prove their local theorems. Indeed, Herman developed a host of new techniques to prove his theorem.

Later, Herman's student Yoccoz improved Herman's result. In his thesis, he showed that if $\rho(f)$ satisfies a Diophantine condition (without restriction on the exponent) and f is C^∞ (resp. analytic), then h is C^∞ (resp. analytic). In his thesis, Herman had proved that if α is Liouville (i.e. it is irrational, but does not satisfy a Diophantine condition), then there exists a C^∞ diffeomorphism f with $\rho(f) = \alpha$ such that h is not even absolutely continuous.

Thus Herman's and Yoccoz's results provided the necessary and sufficient conditions on α for h to be C^∞ when f is C^∞ and $\rho(f) = \alpha$. The analogous result in the analytic case was later obtained by Yoccoz, by entirely different methods.

Herman constructed the first example of a smooth minimal diffeomorphism with positive topological entropy. His example (on a 4-manifold) has a smooth positive invariant density and positive metric entropy for that density.

Herman wrote two volumes on invariant curves of the annulus. Moser's methods can be pushed to show that curves persist under $C^{3+\varepsilon}$ small perturbations (under suitable hypotheses) ; Herman proved they can be destroyed by $C^{3-\varepsilon}$ small perturbations. Using new and deep techniques, he also proved that they persist under C^3 small perturbations. Herman once complained to me that the only person who read the two volumes in their entirety was Yoccoz. It isn't to everyone's taste to read a very difficult proof that curves persist under C^3 small perturbation, when he already knows that they persist under $C^{3+\varepsilon}$ small perturbation! But when it came to understanding a mathematical question, Herman was obstinate : he wanted to understand it completely! However, even if one doesn't want to know why curves persist under C^3 perturbations, one may find much that is interesting in those two volumes.

In a conference in Lyon in 1990, Herman announced spectacular results on symplectic diffeomorphisms. Yoccoz reported on these results in a Bourbaki seminar in 1992, and here we can only mention the highlights of his report : counter-examples to the closing lemma and the quasi-ergodic hypothesis in the class of C^∞ Hamiltonian flows. The latter gives an answer to a very natural interpretation of a question raised by P. and T. Ehrenfest in 1913.

However, as Herman was the first to point out, the real impact of his example is to show that this is the wrong interpretation. His examples occur on non-exact symplectic manifolds. In fact, the periods of the symplectic form must satisfy a Diophantine condition for Herman's construction to work! On the other hand, the physical examples which the Ehrenfests considered all occur on exact symplectic manifolds, in fact, on cotangent bundles. The problems which Herman solved remain open for Hamiltonian flows on exact symplectic manifolds. Nonetheless, Herman's unexpected examples are important, since they show the surprising fact that a global hypothesis is needed for there to be any chance of a positive solution of these problems, even though they don't appear to be of a global nature.

In the middle 90's, Herman solved a fundamental problem in Hamiltonian dynamics. He constructed a compact, connected C^∞ hypersurface $M^{2n-1} \subset \mathbb{R}^{2n}$, $n \geqslant 4$ such that the characteristic flow on M^{2n-1} has no periodic orbits. The same result was obtained independently by Victor Ginzburg, by a different method. (Ginzburg's method works for $n \geqslant 3$.)

For many years, until the end of his life, Herman was working on an ambitious project to generalize KAM theory to situations where the Kolmogorov non-degeneracy hypothesis does not hold. This is very important to do, since many examples of integrable systems in physics are degenerate in the Kolmogorov sense. For example, the Newtonian three body problem is degenerate in the Kolmogorov sense. This is why Arnold's proof of stability in the planar Newtonian three body problem is so difficult. In such situations, one cannot apply the Kolmogorov theorem directly, but nonetheless Herman showed that one can often prove the existence of invariant tori in such situations. Herman prepared a long set of notes, but, unfortunately, nothing of this has been published.

Herman headed his ICM98 article with a quotation from Laplace : « Ce que nous savons est peu de choses ; ce que nous ignorons est immense. » This truly reflects Herman's approach to mathematics. He always had unsolved problems which he wished to discuss. If you solved one of his problems, he was the first to express his admiration. His long list of unsolved problems was one aspect of his success as a thesis advisor, since his students always had interesting things to work on. The many achievements of his students attest to this success.

When Herman began his career, there was little activity in France in the subjects which interested him. Today, there is a great deal of activity, thanks to the school which he founded. Many will miss him.

ACKNOWLEDGEMENT : I wish to thank Bill Veech for a helpful discussion.

Souvenirs de Michel

Jean-Christophe Yoccoz

Automne 1976

Rentré un an plus tôt rue d'Ulm, après une maîtrise à Jussieu perturbée par les derniers soubresauts soixante-huitards et les plaisirs du Quartier Latin, je vais voir à Orsay Jean Cerf, qui a été condisciple de mon père vingt-cinq ans plus tôt à l'École normale. J'ai été séduit l'année précédente par le cours de topologie algébrique de Michel Zisman. Aussi Cerf m'oriente-t-il vers le séminaire de chirurgie animé par Jean Barge, Jean Lannes et Pierre Vogel. En même temps, signe de mon indécision profonde, je suis le cours de Mike Shub sur les systèmes dynamiques hyperboliques, ainsi que celui de Beals sur les E.D.P. J'assiste aussi à quelques séances homériques du séminaire consacré aux travaux de Thurston sur les difféomorphismes des surfaces ; j'y découvre, à la faveur d'empoignades entre Adrien Douady et François Laudenbach, la difficulté d'établir un consensus sur la justesse d'un argument.

Quelques mois plus tard, je retourne voir Cerf : la chirurgie est trop algébrique à mon goût, je n'ai pas trop compris ce qui se passe du côté de chez Thurston, j'ai par contre bien réagi au cours de Shub ; qu'y a-t-il à faire dans cette direction ?

C'est ainsi que j'entends parler pour la première fois de Michel. Bien sûr, il vient de démontrer la conjecture d'Arnold, le théorème global de conjugaison différentiable des difféomorphismes du cercle à des rotations sous une hypothèse diophantienne appropriée sur le nombre de rotation. Mais à l'époque j'ignore même qu'on puisse s'intéresser aux difféomorphismes du cercle. Je prends contact une première fois avec Michel au printemps 77. Mais l'Agrégation approche, et les choses sérieuses sont remises à l'automne suivant.

Automne 1977

Pierre Arnoux et moi sommes formellement les premiers étudiants de Michel, bien qu'Albert Fathi, initialement sous la tutelle de Larry Siebenmann, travaille en fait avec Michel depuis quelques années. Michel nous distribue une demi-douzaine d'articles ; il s'agit d'en choisir un ou deux pour les décortiquer. Ce choix n'est pas sans conséquences sur la suite des opérations. Pierre s'attache à un article sur les échanges d'intervalle ; je porte mon dévolu sur un article de Nancy Kopell traitant des centralisateurs de difféomorphismes de l'intervalle. Un peu plus tard, Michel attire mon attention sur un article de Eddy Zehnder démontrant le théorème de linéarisation de Siegel par la méthode de Nash-Moser. Rétrospectivement, c'est une preuve impressionnante de la profondeur avec laquelle Michel avait déjà à cette époque réfléchi à de très nombreuses