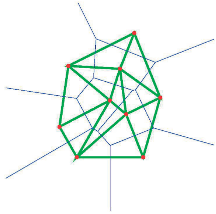
Reconstruction of surfaces



Jean-Daniel Boissonnat

Reconstructing a surface from the knowledge of only some of its points: a problem that one comes across often, be it in geological exploration, in recording archaeological remains, or in medical or industrial imaging.

When we probe the ground in some places to find out the configuration of various geological layers underneath, or when we want to map the sea-bed, the number of points where measurements are made is necessarily finite. The corresponding surfaces need to be reconstructed starting from this limited amount of data. The situation is similar for all computerised imaging systems (scanners, remote-sensors, three-dimensional imaging, etc.) used in medicine, in industry, in archaeology, etc. The starting point is a real object - which can be a part of the human body, a machine part, an archaeological remain, a geological structure, or something else. Instruments can measure this real object only at a certain number of points from which we have to reconstruct the shape of the object virtually. This is the problem of reconstructing surfaces (Figure 1). It thus consists in using a finite number of points to provide a geometrical or a computer representation of the object which will allow us to visualise it on a screen, to store it in the memory of a compu-

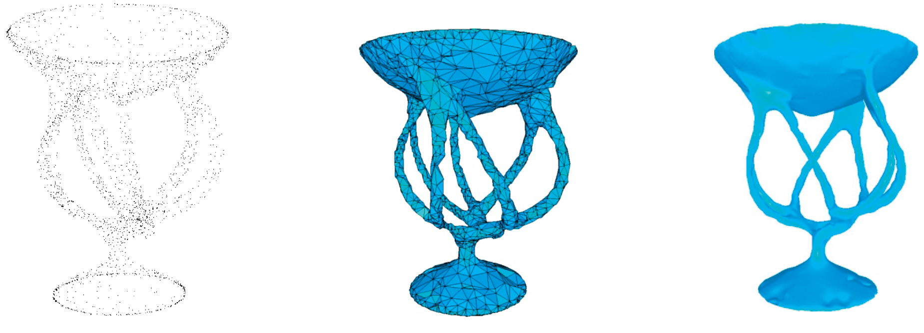


Figure 1. The reconstruction of a surface starting from a sample of its points: this problem arises in various fields.

ter, to easily carry out calculations, even to modify the object or to give instructions by remote control to get a copy tooled. In short, once the form of a real object is digitally recorded with sufficient precision, there are many possibilities for action and calculation

The economic and industrial stakes involved in the problem of surface reconstruction, and its fundamental character from a scientific point of view, have led to many works devoted to it for some twenty years. But it is only very recently that the specialists have formalised the problem in mathematical terms, which enabled them to conceive efficient algorithms furnishing a faithful reconstruction. The results of this so-called algorithmic geometry were transferred very rapidly to industry through the creation of young start-ups (such as *Raindrop Geomagic* in the United States), or by launching new products by the leaders in computer-assisted design or in medical imaging (*Dassault Systèmes, Medical Siemens*).

Voronoï diagrams and Delaunay triangulations, two essential geometric tools

For reconstructing a surface from a mass of sample points, a large majority of algorithms use a central tool in algorithmic geometry: Delaunay triangulation, named after Boris Delone (1890-1980), a Russian mathematician whose name was rendered as Delaunay in French. A Delaunay triangulation is defined in a natural way starting from what is called a Voronoï diagram, after the name of the Ukrainian mathematician Georgi Voronoï (1868-1908). Let us consider a finite set of points in space and call it *E*. The Voronoï diagram of *E* is a division of space into convex cells (shown in blue in Figure 2) where each cell consists of the points of space closer to a certain point of *E* than to any other point of *E*. Cells - they are convex polyhedra - are thus defined in a unique manner.

Now, let us connect by line segments the points of E whose Voronoï cells are adjacent. The set of these segments constitutes the Delaunay triangulation (shown in green in Figure 2) associated to *E*. These structures can be defined in spaces of arbitrary dimension; it is the case dimension three - of the usual space - which is the most interesting one for surface reconstruction. Voronoï diagrams (Figures 2 and 3) are among the main subjects of study in algorithmic geometry, and it is in the 1980's that their relationship with the theory of polytopes (analogues of polyhedra in spaces of dimension higher than three) was

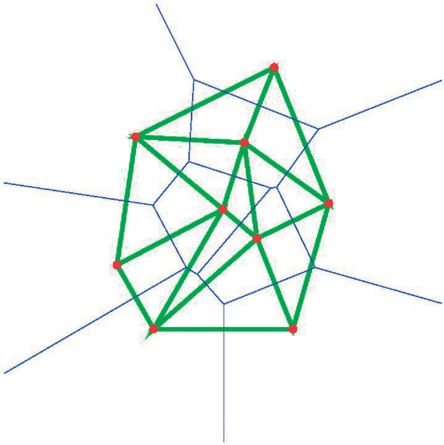


Figure 2. The Voronoï diagram (in blue) and Delaunay triangulation (in green) of a set of points (marked in red). Voronoï diagrams and Delaunay triangulations are fundamental tools in algorithmic geometry.

established. Their study in the context of surface sampling is much more recent.

Why are Voronoï diagrams and Delaunay triangulations interesting ? If *E* is a sample of n points taken on some surface *S*, one can show that the corresponding Voronoï diagram and Delaunay triangulation contain a lot of information about this surface. When the sample is sufficiently dense, it can be shown to provide a precise approximation of the surface. For example, the vector which joins a point *P* of *E* to the most distant vertex of its Voronoï cell is a good approximation of the normal on surface *S* at point *P*.

One should ensure that calculation times remain reasonable, that the algorithms are reliable

Several reconstruction algorithms are now known capable of constructing a surface *S*' which correctly approximates the real surface *S* starting from a finite sample of points on *S*. What's more, the theory of these algorithms allows one to calculate an upper boundary for the difference between *S*' and *S*', a boundary which obviously depends on the sampling density.

As the data sets provided by measuring instruments generally contain several hundreds of thousands - even millions - of points, combinatorial and algorithmic questions play a critical role. It is, for example, important to know if the quantity of calculations which Delaunay triangulations require will remain within a reasonable limit or not. In the most unfavourable cases, the number *T* of calculation steps (i.e., in the final analysis, the computation time) can be quadratic; in other words, *T* is, at worst, proportional to the square of the number of sample points. It is, however, assumed that this situation does not arise in the case of well-sampled surfaces. More precise results were very recently obtained in the case of polyhedric surfaces *S*, i.e. surfaces composed only of polygonal faces: for such surfaces and under weak sampling conditions, the size of the calculation for computing the triangulation is at worst proportional to the number of sample points. The case of smooth surfaces is more delicate; it is currently the object of active research.

The theoretical bounds are not all; it remains to know how to efficiently and rapidly calculate the triangulation from a data set. Many algorithms are known. The more efficient ones are called randomised because they carry out certain random samplings during their execution. The theory of randomised algorithms developed very rapidly in the 1990's

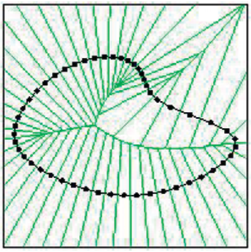


Figure 3. The Voronoï diagram of a set of points on a curve.

and has led to precise analyses validated by experiments. In many cases, and the calculation of the Delaunay triangulation is one of them, the introduction of an element of randomness allows one not to try optimally to solve the worst case (which is very improbable), and has led to simple and very efficient algorithms on the average. One can thus treat samples of 100000 points in some ten seconds (Pentium III with 500 MHz).

If it is important to calculate rapidly, to calculate reliably is even more important. This is a delicate question because, in general, computers only know how to represent numbers with a finite precision (a finite number of decimals). Thus it is impossible to give a representation of numbers having infinitely many decimals such as π or $\sqrt{2}$, which would be at the same time digital and exact. The accumulation of round-off errors can lead to the abnormal behaviour of the programs. Although this behaviour is well known, it is difficult to control, which makes writing and maintaining reliable algorithms very expensive. A significant part of current research in algorithmic geometry is related to these questions and combines the theory of algorithms, formal computation (wherein the computer handles symbols and not explicit numbers) and computer arithmetic. It has already led to the development of software libraries which have made programming easy, efficient and reliable, such as the library CGAL (Computational Geometry Algorithms Library) developed by an international collaboration of universities and research organisations.

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