A NOTE ON INTERSECTIONS OF SIMPLICES

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Abstract. — We provide a corrected proof of [1, Théorème 9] stating that any metrizable infinite-dimensional simplex is affinely homeomorphic to the intersection of a decreasing sequence of Bauer simplices.

Résumé (Sur certaines intersections de simplexes). — Nous exposons une démonstration rectifiée de [1, Théorème 9], montrant ainsi que tout simplexe de Choquet métrisable et de dimension infinie se représente comme intersection d’une suite décroissante de simplexes de Bauer.
1. Introduction

If $X$ is a compact convex subset of a locally convex space over the real numbers, it is called a **Choquet simplex** (briefly **simplex**) if the dual $(A(X))^*$ to the space $A(X)$ of all affine continuous functions is a lattice. If, moreover, the set of all extreme points of $X$ is closed, $X$ is termed a **Bauer simplex** (see [2] for more information on simplices).

The following theorem can be found as [1, Théorème 9]. By $(\ell^1, w^*)$ we mean $\ell^1$ with the topology $\sigma(\ell^1, c_0)$.

**Theorem 1.1.** — Let $X$ be a metrizable infinite-dimensional simplex. Then there exists a decreasing sequence $(T_n)_{n \in \mathbb{N}}$ of Bauer simplices in $(\ell^1, w^*)$ such that $\bigcap_{n=1}^{\infty} T_n$ is affinely homeomorphic to $X$.

Unfortunately, the proof presented in [1] is not entirely correct, since the inclusion $S_{n+1} \cup F_{n+1} \subset (\text{conv}(S_n \cup \{e^{n+1}\})) \cup F_{n+1}$ on page 237 of [1] need not hold in general.

The aim of our note is to indicate how to mend the proof of this theorem.

By [3, Theorem 5.2] (see also [2, Theorem 3.22]), for every metrizable infinite-dimensional simplex $X$ there exists an inverse sequence $(X_n, \varphi_n)_{n \in \mathbb{N}}$ of $(n-1)$-dimensional simplices such that $X$ is affinely homeomorphic to its inverse limit $\lim_{\leftarrow} X_n$. More precisely, every $\varphi_n : X_{n+1} \to X_n$ is an affine continuous surjection and $X$ is affinely homeomorphic to

\[
\{ (x_n) \in \prod_{n=1}^{\infty} X_n : \varphi_n(x_{n+1}) = x_n, n \in \mathbb{N} \}.
\]

Inverse sequences $(X_n, \varphi_n)_{n \in \mathbb{N}}$ and $(Y_n, \psi_n)_{n \in \mathbb{N}}$ are called **equivalent** if there exist affine homeomorphisms $\omega_n : X_n \to Y_n$ such that $\omega_n \circ \varphi_n = \psi_n \circ \omega_{n+1}$, $n \in \mathbb{N}$. Clearly, two equivalent inverse sequences have the same inverse limit up to an affine homeomorphism.

A description of a simplex by an inverse sequence yields a method of representing $X$ by an infinite matrix $A$ that is constructed inductively as follows.

In the first step, let $X_1 = \{ u_1 \}$. Assume now that $n \in \mathbb{N}$ and $\{ u_1^n, \ldots, u_n^n \}$ is the enumeration of vertices of $X_n$ chosen in the $n$-th step.

We choose vertices $\{ u_1^{n+1}, \ldots, u_n^{n+1} \}$ of $X_{n+1}$ such that $\varphi_n(u_i^{n+1}) = u_i^n$, $i = 1, \ldots, n$. If $u_{n+1}^{n+1} \in X_{n+1}$ is the remaining vertex, let $a_{1,n}, \ldots, a_{n,n}$ be positive numbers with $\sum_{i=1}^{n} a_{i,n} = 1$ such that

\[
\varphi_n(u_{n+1}^{n+1}) = \sum_{i=1}^{n} a_{i,n} u_i^n.
\]
Then

\[
A = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\
0 & a_{2,2} & a_{2,3} & \cdots \\
0 & 0 & a_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is the representing matrix of \( X \).

It is not difficult to see that \( A \) is uniquely determined by the inverse sequence \((X_n, \varphi_n)_{n \in \mathbb{N}}\).

Conversely, any such matrix describes a unique inverse sequence of simplices and thus codes a unique metrizable simplex.

We refer the reader to [2], [3], [4] and [5] for detailed information on representing matrices.

We need the following observation based upon [4, Theorem 4.7].

**Proposition 1.2.** — Let \( A \) be a representing matrix for a simplex \( X \). Then there exists a matrix \( B = \{b_{i,n}\}_{1 \leq i \leq n} \) representing \( X \) such that \( b_{i,n} > 0 \) for all \( 1 \leq i \leq n \) and \( n = 1, 2, \ldots \).

**Proof.** — It follows from [4, Theorem 4.7] that two matrices \( A \) and \( B \) represent the same simplex if \( \sum_{n=1}^{\infty} \sum_{i=1}^{n} |a_{i,n} - b_{i,n}| < \infty \). Thus it is enough to slightly perturb the coefficients of \( A \) to get the required matrix \( B \).

\[\square\]

2. Proof of Theorem 1.1

We recall some notation from [1]. Let \( e^n, n \in \mathbb{N} \), denote the standard basis vectors in \( \ell^1 \) and let \( e^0 = 0 \).

For \( n \in \mathbb{N} \), let \( E_n = \text{conv} \{e^0, \ldots, e^{n-1}\} \) and let \( P_n : \ell^1 \to \ell^1 \) be the natural projection on the space spanned by vectors \( e^0, \ldots, e^{n-1} \), precisely

\[
P_n : (x_1, x_2, \ldots) \mapsto (x_1, \ldots, x_{n-1}, 0, 0, \ldots), \quad (x_1, x_2, \ldots) \in \ell^1.
\]

In particular, \( P_1 \) maps \( \ell^1 \) onto \( e^0 \).

We state an easy observation needed in the proof of Proposition 2.2.

**Lemma 2.1.** — Let \( X \) be a finite-dimensional simplex in a vector space \( E \) containing \( 0 \) and \( x \) be a vector not contained in the linear span of \( X \).

Then for any \( y \) in the relative interior of \( X \) there exists \( \varepsilon > 0 \) such that \( y + \varepsilon x \in \text{conv}(X \cup \{x\}) \).
Proof. — If \( y \) is in the relative interior of \( X \) and \( 0 \in X \), there exists \( \varepsilon \in (0, 1) \) such that 
\[
(1 - \varepsilon)^{-1} y \in X.
\]
Then
\[
y + \varepsilon x = (1 - \varepsilon) \frac{y}{1 - \varepsilon} + \varepsilon x \in \text{conv}(X \cup \{x\}),
\]
which finishes the proof. \( \square \)

Now we start with the proof of Theorem 1.1. Given a metrizable simplex \( X \), Proposition 1.2 provides an inverse sequence \((X_n, \varphi_n)_{n \in \mathbb{N}}\) such that \( X \) is its inverse limit and the associated representing matrix \( A \) has all entries \( a_{i,n} > 0 \) for all \( n \in \mathbb{N} \) and \( 1 \leq i \leq n \).

**Proposition 2.2.** — Let \( X \) be a metrizable infinite-dimensional simplex with a representing matrix \( A \) such that \( a_{i,n} > 0 \) for all \( n \in \mathbb{N} \) and \( 1 \leq i \leq n \).

Let \((X_n, \varphi_n)_{n \in \mathbb{N}}\) be the inverse sequence associated with \( A \).

Then there exist \((n - 1)\)-dimensional simplices \( S_n \subset \ell^1 \), \( n \in \mathbb{N} \), such that

(i) \( S_n \subset E_n \), \( n \in \mathbb{N} \),
(ii) \( S_n \) is a face of \( S_m \), \( n < m \),
(iii) \( P_n S_m = S_n \), \( n < m \),
(iv) \( S_{n+1} \subset \text{conv}(S_n \cup \{e^n\}) \), \( n \in \mathbb{N} \),
(v) the inverse sequences \((X_n, \varphi_n)_{n \in \mathbb{N}}\) and \((S_n, P_n)_{n \in \mathbb{N}}\) are equivalent.

Proof. — We construct inductively simplices \( S_n \) together with mappings \( \omega_n : X_n \rightarrow S_n \), \( n \in \mathbb{N} \), observing that the resulting inverse sequence is equivalent to the original one.

We start the construction by setting \( S_1 = E_1 = \{e^0\} \) and \( S_2 = E_2 = \text{conv}\{e^0, e^1\} \). Let \( \omega_1 : X_1 \rightarrow S_1 \) and \( \omega_2 : X_2 \rightarrow S_2 \) be the obvious affine homeomorphisms.

We assume that the construction has been completed up to the \( n\)-th stage. If \( \omega_n : X_n \rightarrow S_n \) is the affine homeomorphism guaranteed by the inductive assumption and \( \{u_0^n, \ldots, u_n^n\} \) are the vertices of \( X_n \), then \( \{\omega_n(u_0^n), \ldots, \omega_n(u_n^n)\} \) are the vertices of \( S_n \).

Let \( \{u_1^{n+1}, \ldots, u_{n+1}^{n+1}\} \) be the vertices of \( X_{n+1} \) that are mapped by \( \varphi_n \) onto the vertices \( \{u_1^n, \ldots, u_n^n\} \) of \( X_n \) and let \( u_{n+1}^{n+1} \) be the remaining vertex mapped onto the point \( \sum_{i=1}^n a_{i,n} u_i^n \).

Since all numbers \( a_{1,n}, \ldots, a_{n,n} \) are strictly positive, the point
\[
\omega_n(\varphi_n(u_{n+1}^{n+1})) = \sum_{i=1}^n a_{i,n} \omega_n(u_i^n)
\]
is contained in the relative interior of \( S_n \). By Lemma 2.1, there exists \( \varepsilon > 0 \) such that
\[
\omega_n(\varphi_n(u_{n+1}^{n+1})) + \varepsilon e^n \in \text{conv}(S_n \cup \{e^n\}).
\]

(2)
By defining

\( S_{n+1} = \text{conv}(S_n \cup \{ \omega_n(\varphi_n(u^{n+1}_n)) + \varepsilon e^n \}) \)

we get an \( n \)-simplex with vertices

\( \{ \omega_n(u^1_n), \ldots, \omega_n(u^n_n), \omega_n(\varphi_n(u^{n+1}_n)) + \varepsilon e^n \} \).

We define \( \omega_{n+1} : X_{n+1} \to S_{n+1} \) by conditions

\[
\begin{align*}
\omega_{n+1}(u^{n+1}_i) &= \omega_n(\varphi_n(u^{n+1}_i)), & i &= 1, \ldots, n, \\
\omega_{n+1}(u^{n+1}_n) &= \omega_n(\varphi_n(u^{n+1}_n)) + \varepsilon e^n.
\end{align*}
\]

By (2) and (3) and the inductive assumption,

\( S_{n+1} \subset \text{conv}(S_n \cup \{ e^n \}) \subset E_{n+1} \).

Further, \( S_n \) is a face of \( S_{n+1} \), \( P_n S_{n+1} = S_n \) and \( \omega_n \circ \varphi_n = P_n \circ \omega_{n+1} \).

Thus conditions (i)–(iv) are satisfied and the mappings \( \omega_n, n \in \mathbb{N} \), show that the sequences \( (X_n, \varphi_n) \) and \( (S_n, P_n) \) are equivalent.

This finishes the proof.

The rest of the proof Theorem 1.1 can proceed as in [1]. To clarify what is going on, we give two more propositions. The proof of Theorem 1.1 follows immediately from them.

**Proposition 2.3.** — Let \( S_n, n \in \mathbb{N} \), be weak* compact convex subsets of \( \ell^1 \) satisfying conditions (i), (ii') and (iii), where (i) and (iii) are conditions from Proposition 2.2 and

\( (ii') \) \( S_n \subset S_m \) for \( n \leq m \).

Then the inverse limit of the inverse sequence \( (S_n, P_n)_{n \in \mathbb{N}} \) is affinely homeomorphic to the closure of \( \bigcup_{n=1}^\infty S_n \) in the weak* topology.

**Proof.** — Let \( Y \) denote the weak*-closure of \( \bigcup_{n=1}^\infty S_n \), and let \( X \) be the inverse limit \( \lim_{\leftarrow} S_n \) represented in the form given by the formula (1). An affine homeomorphism \( \varphi : Y \to X \) can be defined by the equation

\[
\varphi(y) = (P_n(y))_{n \in \mathbb{N}}, \quad y \in Y.
\]

To see that \( \varphi \) is well defined, note that by (ii') and (iii) we have \( P_n(y) \in S_n \) whenever \( y \in \bigcup_{n=1}^\infty S_n \), and hence, by the weak*-continuity of \( P_n : \ell^1 \to \ell^1 \), that \( P_n(y) \in S_n \) for all \( y \in Y \). Moreover, \( \varphi \) is clearly affine, continuous and one-to-one. To see that \( \varphi \) is onto, choose any \( x = (x_n)_{n \in \mathbb{N}} \in X \). Let \( y \in \mathbb{R}^\mathbb{N} \) have as \( n \)-th coordinate \( y_n \) the \( n \)-th coordinate of the vector \( x_{n+1} \). Then \( (y_1, \ldots, y_n, 0, \ldots) \in S_n \) for each \( n \in \mathbb{N} \), therefore \( y \in \ell^1 \) by (i), and so \( y \in Y \). Moreover, clearly \( \varphi(y) = x \). This completes the proof.
Proposition 2.4. — Let \((S_n)_{n \in \mathbb{N}}\) be a sequence of simplices in \(\ell^1\) satisfying conditions (i)–(iv) of Proposition 2.2.

Set

\[ F_n = \operatorname{conv}\{e^0, e^n, e^{n+1}, \ldots\}, \quad n \in \mathbb{N}, \]

where the bar denotes norm-closure, and

\[ T_n = \operatorname{conv}(S_n \cup F_n), \quad n \in \mathbb{N}. \]

Then \((T_n)_{n \in \mathbb{N}}\) is a decreasing sequence of Bauer simplices in \((\ell^1, w^*\) whose intersection is the weak*-closure of \(\bigcup_{n=1}^{\infty} S_n\).

Proof. — It is clear that both \(F_n\) and \(S_n\) are Bauer simplices in \((\ell^1, w^*)\). Thus \(T_n\) is a Bauer simplex in \((\ell^1, w^*)\) as well. Moreover,

\[ S_{n+1} \cup F_{n+1} \subset (\operatorname{conv}(S_n \cup \{e^n\})) \cup F_{n+1} \]

\[ \subset \operatorname{conv}(S_n \cup F_n), \]

and hence \(T_{n+1} \subset T_n\) for \(n \in \mathbb{N}\).

It remains to prove the final equality.

Set \(T = \bigcap_{n=1}^{\infty} T_n\) and denote by \(Y\) the weak*-closure of \(\bigcup_{n=1}^{\infty} S_n\). Let \(n \in \mathbb{N}\) be arbitrary. Then for each \(m \geq n\) we have \(S_n \subset S_m \subset T_m\). Thus \(S_n \subset T\). It follows that \(Y \subset T\).

To see the converse inclusion, take any \(x \in T\). For each \(n \in \mathbb{N}\) we have \(x \in T_n, 0 \in S_n\), and hence \(P_n(x) \in S_n\). But the sequence \((P_n(x))_{n \in \mathbb{N}}\) is weak* convergent to \(x\), so \(x \in Y\).

Finally, Theorem 1.1 follows immediately by combining Propositions 1.2, 2.2, 2.3 and 2.4.

Remark 2.5. — We note that it is not essential that we work in the space \((\ell^1, w^*)\). The norm structure of this space is used only in the definition of \(F_n\), and can be replaced there by weak*-closure. So, it would be possible (and, perhaps, more natural) to work in the locally convex space \(\mathbb{R}^N\) equipped with the pointwise topology. Anyway, we decided to keep the setting from [1].

Bibliography


