WHEN IS A RIESZ DISTRIBUTION
A COMPLEX MEASURE?

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ABSTRACT. — Let $R_\alpha$ be the Riesz distribution on a simple Euclidean Jordan algebra, parametrized by $\alpha \in \mathbb{C}$. I give an elementary proof of the necessary and sufficient condition for $R_\alpha$ to be a locally finite complex measure (= complex Radon measure).

RÉSUMÉ (Une distribution de Riesz, quand est-elle mesure complexe ?)

Soit $R_\alpha$ la distribution de Riesz sur une algèbre de Jordan euclidienne simple, paramétrisée par $\alpha \in \mathbb{C}$. Je donne une démonstration élémentaire de la condition nécessaire et suffisante pour que $R_\alpha$ soit une mesure complexe localement finie (= mesure de Radon complexe).

1. Introduction

In the theory of harmonic analysis on Euclidean Jordan algebras (or equivalently on symmetric cones) [12], a central role is played by the *Riesz distributions* $R_\alpha$, which are tempered distributions that depend analytically on a...
parameter $\alpha \in \mathbb{C}$. One important fact about the Riesz distributions is the necessary and sufficient condition for positivity, due to Gindikin [13]:

Theorem 1.1. — [12, Theorem VII.3.1] Let $V$ be a simple Euclidean Jordan algebra of dimension $n$ and rank $r$, with $n = r + \frac{d}{2} r (r - 1)$. Then the Riesz distribution $R_\alpha$ on $V$ is a positive measure if and only if $\alpha = 0, \frac{d}{2}, \ldots, (r - 1) \frac{d}{2}$ or $\alpha > (r - 1) \frac{d}{2}$.

The “if” part is fairly easy, but the “only if” part is reputed to be deep [13, 12, 20].

The purpose of this note is to give a completely elementary proof of the “only if” part of Theorem 1.1, and indeed of the following strengthening:

Theorem 1.2. — Let $V$ be a simple Euclidean Jordan algebra of dimension $n$ and rank $r$, with $n = r + \frac{d}{2} r (r - 1)$. Then the Riesz distribution $R_\alpha$ on $V$ is a locally finite complex measure [= complex Radon measure] if and only if $\alpha = 0, \frac{d}{2}, \ldots, (r - 1) \frac{d}{2}$ or $\Re \alpha > (r - 1) \frac{d}{2}$.

This latter result is also essentially known [18, Lemma 3.3], but the proof given there requires some nontrivial group theory.

The idea of the proof of Theorem 1.2 is very simple: A distribution defined on an open subset $\Omega \subset \mathbb{R}^n$ by a function $f \in L^1_{\text{loc}}(\Omega)$ can be extended to all of $\mathbb{R}^n$ as a locally finite complex measure only if the function $f$ is locally integrable also at the boundary of $\Omega$ (Lemma 2.1); furthermore, this fact survives analytic continuation in a parameter (Proposition 2.3). In the case of the Riesz distribution $R_\alpha$, a simple computation using its Laplace transform (Lemma 3.4) plus a bit of extra work (Lemma 3.5) allows us to determine the allowed set of $\alpha$, thereby proving Theorem 1.2.

Theorem 1.2 thus states a necessary and sufficient condition for $R_\alpha$ to be a distribution of order 0. It would be interesting, more generally, to determine the order of the Riesz distribution $R_\alpha$ for each $\alpha \in \mathbb{C}$.

It would also be interesting to know whether this approach is powerful enough to handle the multiparameter Riesz distributions $R_{\alpha}$ with $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$ [12, Theorem VII.3.2] and/or the Riesz distributions on homogeneous cones that are not symmetric (i.e. not self-dual) and hence do not arise from a Euclidean Jordan algebra [13, 20].

In an Appendix I comment on a beautiful but little-known elementary proof of Theorem 1.1 — which does not extend, however, to Theorem 1.2 — due to Shanbhag [27] and Casalis and Letac [9].

(1) The set of values of $\alpha$ described in Theorem 1.1 is the so-called Wallach set [29, 30, 21, 10, 11, 12].
2. A general theorem on distributions

We assume a basic familiarity with the theory of distributions [26, 19] and recall some key notations and facts.

For each open set \( \Omega \subseteq \mathbb{R}^n \), we define the space \( \mathcal{D}(\Omega) \) of \( C^\infty \) functions having compact support in \( \Omega \), the corresponding space \( \mathcal{D}'(\Omega) \) of distributions, and the space \( \mathcal{D}^k(\Omega) \) of distributions of order \( \leq k \). In particular, the space \( \mathcal{D}^0(\Omega) \) consists of the distributions that are given locally (i.e. on every compact subset of \( \Omega \)) by a finite complex measure.

Let \( f : \Omega \to \mathbb{C} \) be a measurable function, and extend it to all of \( \mathbb{R}^n \) by setting \( f \equiv 0 \) outside \( \Omega \). We say that \( f \in L^1_{\text{loc}}(\Omega) \) if, for every \( x \in \Omega \), \( f \) is (absolutely) integrable on some neighborhood of \( x \). Any \( f \in L^1_{\text{loc}}(\Omega) \) defines a distribution \( T_f \in \mathcal{D}'(\Omega) \) by

\[
T_f(\varphi) = \int \varphi(x) f(x) \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega).
\]

We are interested in knowing under what circumstances the distribution \( T_f \in \mathcal{D}'(\Omega) \) can be extended to a distribution \( \tilde{T}_f \in \mathcal{D}'(\mathbb{R}^n) \), i.e. one that is locally everywhere on \( \mathbb{R}^n \) a finite complex measure.

**Lemma 2.1.** — Let \( f : \Omega \to \mathbb{C} \) be in \( L^1_{\text{loc}}(\Omega) \), and let \( T_f \in \mathcal{D}'(\Omega) \) be the corresponding distribution. Then the following are equivalent:

(a) \( f \in L^1_{\text{loc}}(\Omega) \), i.e. for every \( x \in \Omega \), \( f \) is integrable on some neighborhood of \( x \).(2)

(b) There exists a distribution \( \tilde{T}_f \in \mathcal{D}'(\mathbb{R}^n) \) that extends \( T_f \) and is supported on \( \overline{\Omega} \).

(c) There exists a distribution \( \tilde{T}_f \in \mathcal{D}'(\mathbb{R}^n) \) that extends \( T_f \).

**Proof.** — (a) \( \implies \) (b): It suffices to define \( \tilde{T}_f(\varphi) = \int_{\Omega} \varphi(x) f(x) \, dx \) for all \( \varphi \in \mathcal{D}(\mathbb{R}^n) \).

(b) \( \implies \) (c) is trivial.

(c) \( \implies \) (a): By hypothesis, for every \( x \in \partial \Omega \) and every compact neighborhood \( K \ni x \), there exists a finite complex measure \( \mu_K \) supported on \( K \) such that \( \tilde{T}_f(\varphi) = \int \varphi \, d\mu_K \) for every \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) with support in \( K \). But since \( \tilde{T}_f \) extends \( T_f \), the restriction of \( \mu_K \) to every compact subset of \( K \cap \Omega \) must coincide with the measure \( f(x) \, dx \). Since \( K \cap \Omega \) is \( \sigma \)-compact, this implies that \( \int |f(x)| \, dx = |\mu_K|(K \cap \Omega) < \infty \), so that \( f \) is integrable in a neighborhood of \( x \).

\( \Box \)

(2) Since this has already been assumed for \( x \in \Omega \), the content of hypothesis (a) is that it should hold also for \( x \in \partial \Omega \).
We now extend this idea to allow for analytic dependence on a parameter. Let \( \Omega \) be an open set in \( \mathbb{R}^n \), let \( D \) be a connected open set in \( \mathbb{C}^m \), and let \( F : \Omega \times D \to \mathbb{C} \) be a continuous function such that \( F(x, \cdot) \) is analytic on \( D \) for each \( x \in \Omega \). Then, for each \( \lambda \in D \), define

\[
T_\lambda(\varphi) = \int \varphi(x) F(x, \lambda) \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega).
\]

**Lemma 2.2.** — With \( F \) as above, the map \( \lambda \mapsto T_\lambda \) is analytic from \( D \) into \( \mathcal{D}'(\Omega) \) in the sense that \( \lambda \mapsto T_\lambda(\varphi) \) is analytic for all \( \varphi \in \mathcal{D}(\Omega) \).

**Proof.** — This is an immediate consequence of the hypotheses on \( F \) together with standard facts about scalar-valued analytic functions in \( \mathbb{C} \) (either Morera’s theorem or the Cauchy integral formula) and \( \mathbb{C}^m \) (e.g. the weak form of Hartogs’ theorem).

**Remark.** — Weak analyticity in the sense used here is actually equivalent to strong analyticity: see e.g. [15, pp. 37–39, Théorème 1 and Remarque 1] [5, Théoremes 3.1 and 3.2] [14, Théorème 1]. Indeed, our hypothesis on \( F \) is equivalent to the even stronger statement that the map \( \lambda \mapsto F(\cdot, \lambda) \) is analytic from \( D \) into the space \( C^0(\Omega) \) of continuous functions on \( \Omega \), equipped with the topology of uniform convergence on compact subsets [15, p. 41, example (a)]. But we do not need any of these facts; weak analyticity is enough for our purposes.

Putting together these two lemmas, we obtain:

**Proposition 2.3.** — Let \( F \) be as above, let \( D_0 \subseteq D \) be a nonempty open set, and let \( \lambda \mapsto \tilde{T}_\lambda \) be a (weakly) analytic map of \( D \) into \( \mathcal{D}'(\mathbb{R}^n) \) such that \( \tilde{T}_\lambda \) extends \( T_\lambda \) for each \( \lambda \in D_0 \). Then, for each \( \lambda \in D \), we have:

(a) \( \tilde{T}_\lambda \) extends \( T_\lambda \).

(b) If \( \tilde{T}_\lambda \in \mathcal{D}'(\mathbb{R}^n) \), then \( F(\cdot, \lambda) \in L^1_{\text{loc}}(\Omega) \).

**Proof.** — (a) This is immediate by analytic continuation: for each \( \varphi \in \mathcal{D}(\Omega) \), both \( \tilde{T}_\lambda(\varphi) \) and \( T_\lambda(\varphi) \) are (by hypothesis and Lemma 2.2, respectively) analytic functions of \( \lambda \) on \( D \) that coincide on \( D_0 \), therefore they must coincide on all of \( D \).

(b) This is immediate from (a) together with Lemma 2.1.

We shall apply this setup with \( F(x, \lambda) = f(x)^\lambda \) where \( f : \Omega \to (0, \infty) \) is a continuous function; in fact, we shall take \( f \) to be a polynomial.
Remark. — Let $P$ be a polynomial that is strictly positive on $\Omega$ and vanishes on $\partial \Omega$, and define for $\text{Re} \lambda > 0$ a tempered distribution $\mathcal{D}_\lambda \in \mathcal{S}'(\mathbb{R}^n)$ by the formula

$$\mathcal{D}_\lambda(\varphi) = \int_{\Omega} P(x)^\lambda \varphi(x) \, dx \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Then $\mathcal{D}_\lambda$ is a tempered-distribution-valued analytic function of $\lambda$ on the right half-plane, and it is a deep result of Atiyah, Bernstein and S.I. Gelfand [3, 1, 2, 4] that $\mathcal{D}_\lambda$ can be analytically continued to the whole complex plane as a meromorphic function of $\lambda$ with poles on a finite number of arithmetic progressions. It is important to note that our Proposition 2.3 does not rely on this deep result; rather, it says that whenever such an analytic continuation exists (however it may be constructed), the analytically-continued distribution $\mathcal{D}_\lambda$ can be a complex measure only if $P^\lambda \in L^1_{\text{loc}}(\Omega)$.

3. Application to Riesz distributions

We refer to the book of Faraut and Korányi [12] for basic facts about symmetric cones and Jordan algebras. Let $V$ be a simple Euclidean (real) Jordan algebra of dimension $n$ and rank $r$, with Peirce subspaces $V_{ij}$ of dimension $d$; recall that $n = r + \frac{d}{2}r(r-1)$. We denote by $(x|y) = \text{tr}(xy)$ the inner product on $V$, where $\text{tr}$ is the Jordan trace and $xy$ is the Jordan product. Let $\Omega \subset V$ be the positive cone (i.e. the interior of the set of squares in $V$, or equivalently the set of invertible squares in $V$); it is self-dual, i.e. $\Omega^* = \Omega$. We denote by $\Delta(x) = \det(x)$ the Jordan determinant on $V$: it is a homogeneous polynomial of degree $r$ on $V$, which is strictly positive on $\Omega$ and vanishes on $\partial \Omega$, and which satisfies [12, Proposition III.4.3]

$$\Delta(gx) = \text{Det}(g)^{r/n} \Delta(x) \quad \text{for } g \in G, \ x \in V,$$

where $G$ denotes the identity component of the linear automorphism group of $\Omega$ [it is a subgroup of $GL(V)$] and Det denotes the determinant of an endomorphism. We then have the following fundamental Laplace-transform formula:

**Theorem 3.1.** — [12, Corollary VII.1.3] For $y \in \Omega$ and $\text{Re} \alpha > (r-1)\frac{d}{2} = \frac{r}{2} - 1$, we have

$$\int_{\Omega} e^{-(x|y)} \Delta(x)^{\alpha - \frac{d}{2}} \, dx = \Gamma(\alpha) \Delta(y)^{-\alpha}$$

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where

\[
\Gamma_\Omega(\alpha) = (2\pi)^{(n-r)/2} \prod_{j=0}^{r-1} \Gamma\left(\alpha - j d / 2\right) .
\]

Thus, for Re \( \alpha > (r-1)d/2 \), the function \( \Delta(x)^{\alpha - \frac{n}{2}} / \Gamma_\Omega(\alpha) \) is locally integrable on \( \Omega \) and polynomially bounded, and so defines a tempered distribution \( R_\alpha \) on \( V \) by the usual formula

\[
R_\alpha(\varphi) = \frac{1}{\Gamma_\Omega(\alpha)} \int_\Omega \varphi(x) \Delta(x)^{\alpha - \frac{n}{2}} \, dx \quad \text{for } \varphi \in \mathcal{S}(V) .
\]

Using (5), a beautiful argument — which is a special case of Bernstein’s general method for analytically continuing distributions of the form \( \rho^\alpha_\Omega \) [2, 4] — shows that the Riesz distributions \( R_\alpha \) can be analytically continued to the whole complex \( \alpha \)-plane:

**Theorem 3.2.** — [12, Theorem VII.2.2 et seq.] The distributions \( R_\alpha \) can be analytically continued to the whole complex \( \alpha \)-plane as a tempered-distribution-valued entire function of \( \alpha \). Furthermore, the distributions \( R_\alpha \) have the following properties:

1. \( R_0 = \delta \)
2. \( R_\alpha * R_\beta = R_{\alpha+\beta} \)
3. \( \Delta(\partial / \partial x) R_\alpha = R_{\alpha-1} \)
4. \( \Delta(x) R_\alpha = \prod_{j=0}^{r-1} \left( \alpha - j \frac{d}{2} \right) R_{\alpha+1} \)

(here \( \delta \) denotes the Dirac measure at 0) and

\[
R_\alpha(\varphi \circ g^{-1}) = \det(g)^{\alpha r/n} R_\alpha(\varphi) \quad \text{for } g \in G, \varphi \in \mathcal{S}(V)
\]

(in particular, \( R_\alpha \) is homogeneous of degree \( \alpha r - n \)). Finally, the Laplace transform of \( R_\alpha \) is

\[
(\mathcal{L}R_\alpha)(y) = \Delta(y)^{-\alpha}
\]

for \( y \) in the complex tube \( \Omega + iV \).

The property (8d) is not explicitly stated in [12], but for Re \( \alpha > (r-1)d/2 \) it is an immediate consequence of (6)/(7), and then for other values of \( \alpha \) it follows by analytic continuation (see also [18, Proposition 3.1(iii) and Remark 3.2]). Likewise, the property (9) is not explicitly stated in [12], but for Re \( \alpha > (r-1)d/2 \) it is an immediate consequence of (4)/(7), and then for other values of \( \alpha \) it follows by analytic continuation (see also [18, Proposition 3.1(i)]). It follows
from (8a,b) that the distributions $\mathcal{R}_\alpha$ are all nonzero; and it follows from this and (9) that $\mathcal{R}_\alpha \neq \mathcal{R}_\beta$ whenever $\alpha \neq \beta$.

It is fairly easy to find a sufficient condition for the Riesz distributions to be a positive measure:

**Proposition 3.3** ([12, Proposition VII.2.3], see also [18, Section 3.2], [21, 6])

(a) For $\alpha = k\frac{d}{2}$ with $k = 0, 1, \ldots, r-1$, the Riesz distribution $\mathcal{R}_\alpha$ is a positive measure that is supported on the set of elements of $\Omega$ of rank exactly $k$ (which is a subset of $\partial \Omega$).

(b) For $\alpha > (r-1)\frac{d}{2}$, the Riesz distribution $\mathcal{R}_\alpha$ is a positive measure that is supported on $\Omega$ and given there by a density (with respect to Lebesgue measure) that lies in $L^1_{\text{loc}}(\Omega)$.

The interesting and nontrivial fact (Theorem 1.1 above) is that the converse of Proposition 3.3 is also true: the foregoing values of $\alpha$ are the only ones for which $\mathcal{R}_\alpha$ is a positive measure. Here I shall use Proposition 2.3 together with the Laplace-transform formula (5)/(10) to provide an alternate and extremely elementary proof of the stronger converse result stated in Theorem 1.2.

**Lemma 3.4.** — $\Delta^\lambda \in L^1_{\text{loc}}(\Omega)$ if and only if $\text{Re}(\lambda) > -1$; or in other words, $\Delta^{\alpha - \frac{d}{2}} \in L^1_{\text{loc}}(\Omega)$ if and only if $\text{Re}(\alpha) > (r-1)\frac{d}{2} = \frac{n}{r} - 1$.

**Proof.** — Since $|\Delta(x)|^\lambda = \Delta(x)^{\text{Re}\lambda}$, it suffices to consider real values of $\lambda$.

For $\lambda > -1$ [i.e. $\alpha > (r-1)\frac{d}{2}$], fix any $y \in \Omega$: the fact that the integral (5) is convergent, together with the fact that $x \mapsto e^{+(x|y)}$ is locally bounded, implies that $\Delta^\lambda \in L^1_{\text{loc}}(\Omega)$.

Now consider $\lambda = -1$: again fix any $y \in \Omega$, and let $\mu = \inf_{x \in \mathbb{C}, \|x\| = 1} (x|y) > 0$ where $\| \cdot \|$ is any norm on $V$. Choose $R > 0$ such that $|\Delta(x)| \leq 1$ whenever $\|x\| \leq R$. Then

$$\int_{x \in \Omega, \|x\| \leq R} e^{-(x|y)} \Delta(x)^{-1} dx = \lim_{\lambda \uparrow -1} \int_{x \in \Omega, \|x\| \leq R} e^{-(x|y)} \Delta(x) \lambda dx$$

by the monotone convergence theorem. We now proceed to obtain a lower bound on

$$M_\lambda := \int_{x \in \Omega, \|x\| \leq R} e^{-(x|y)} \Delta(x)^\lambda dx.$$
For any $\beta \geq 1$, we have

\begin{align}
(13a) \quad & \int_{\frac{\beta}{2} \leq \|x\| \leq \beta R} e^{-\beta(x|y)} \Delta(x)^\lambda \, dx = \beta^{\alpha+r\lambda} \int_{\frac{\beta}{2} \leq \|x\| \leq R} e^{-\beta(x|y)} \Delta(x)^\lambda \, dx \\
& \leq \beta^{\alpha+r\lambda} e^{-\beta(\frac{\beta}{2})} \mu \int_{\frac{\beta}{2} \leq \|x\| \leq R} e^{-\beta(x|y)} \Delta(x)^\lambda \, dx \\
(13b) \quad & \leq \beta^{\alpha+r\lambda} e^{-\beta(\frac{\beta}{2})} \mu M_A \\
(13c) \quad & \leq \beta^{\alpha+r\lambda} e^{-\beta(\frac{\beta}{2})} \mu M_A
\end{align}

where the first equality used the homogeneity of $\Delta$. Now sum this over $\beta = 2^k$ ($k = 1, 2, 3, \ldots$); the sum is convergent, and we conclude that

\begin{align}
(14) \quad & \int_{x \in \Omega} e^{-\beta(x|y)} \Delta(x)^\lambda \, dx \leq CM_A
\end{align}

for a universal constant $C < \infty$ that is independent of $\lambda$ for $-1 < \lambda \leq 0$. Since $(5)$ tells us that

\begin{align}
(15) \quad & \lim_{\lambda \to -1} \int_{x \in \Omega} e^{-\beta(x|y)} \Delta(x)^\lambda \, dx = +\infty
\end{align}

due to the pole of the gamma function at $\alpha = (r - 1) \frac{d}{2}$, we conclude that $\lim M_A = +\infty$ as well. Therefore

\begin{align}
(16) \quad & \int_{\|x\| \leq R} e^{-\beta(x|y)} \Delta(x)^{-1} \, dx = +\infty,
\end{align}

which proves that $\Delta^{-1} \notin L^1_{\text{loc}}(\Omega)$.

Since $\Delta$ is locally bounded, it also follows that $\Delta^\lambda \notin L^1_{\text{loc}}(\Omega)$ for $\lambda < -1$. \qed

We shall also need a uniqueness result related to Proposition 3.3(a). If $\mu$ is a locally finite complex measure on $V$, we say that $\mu$ is $G$-relatively invariant with exponent $\kappa$ in case

\begin{align}
(17) \quad & \mu(gA) = \text{Det}(g)^{\kappa} \mu(A) \quad \text{for } g \in G, A \text{ compact } \subseteq V.
\end{align}

In particular, every such $\mu$ is $G \cap SL(V)$-invariant, i.e.

\begin{align}
(18) \quad & \mu(gA) = \mu(A) \quad \text{for } g \in G \cap SL(V), A \text{ compact } \subseteq V.
\end{align}
Now define $\Omega_k = \{ x \in \mathbb{R} : \text{rank}(x) = k \}$, so that $\partial \Omega = \bigcup_{k=0}^{r-1} \Omega_k$ and $\Omega = \Omega_r$. We then have the following result, which seems to be of some interest in its own right:

**Lemma 3.5.** — (a) The group $G \cap SL(V)$ acts transitively on each set $\Omega_k$ ($0 \leq k \leq r-1$).

(b) Let $\mu$ be a locally finite complex measure that is supported on $\Omega_k$ ($0 \leq k \leq r-1$) and is $G \cap SL(V)$-invariant. Then $\mu$ is a multiple of $R_{kd/2}$.

(c) Let $\mu$ be a locally finite complex measure that is supported on $\partial \Omega$ and is $G$-relatively invariant with some exponent $\kappa$. Then there exists $k \in \{0,1,\ldots,r-1\}$ such that $\mu$ is a multiple of $R_{kd/2}$ (and hence $\kappa = kd/2n$ if $\mu \neq 0$).

**Proof.** — (a) Fix a Jordan frame $c_1, \ldots, c_r$, and let $V = \bigoplus_{1 \leq i \leq r} V_{ij}$ be the corresponding orthogonal Peirce decomposition [12, Theorem IV.2.1]. Then, for $\lambda > 0$, let $M_\lambda = P(c_1 + \cdots + c_{r-1} + \lambda c_r) \in GL(V)$, where $P$ is the quadratic representation [12, p. 32]. From [12, p. 32 and Theorem IV.2.1(ii)] we see that $M_\lambda$ acts as multiplication by $\lambda^2$ on the space $V_{rr}$, as multiplication by $\lambda$ on the spaces $V_{ir}$ with $1 \leq i \leq r-1$, and as the identity on the other subspaces. (3)

We have $M_\lambda \in G$ [12, Proposition III.2.2] and $\det(M_\lambda) = \lambda^{(r-1)d+2} = \lambda^{2n/r}$.

Now write $e_k = c_1 + \cdots + c_r$. By construction we have $M_\lambda e_k = e_k$ for $0 \leq k \leq r-1$. Now, we know [12, Proposition IV.3.1] that $\Omega_k = G e_k$, so that for any $x \in \Omega_k$ there exists $g \in G$ such that $x = ge_k$. Therefore, if we set $\lambda = \det(g)^{-r/2n}$, we have $x = gM_\lambda e_k$ with $gM_\lambda \in G \cap SL(V)$.

(b) follows from (a) and Proposition 3.3(a) together with a standard result about the uniqueness of invariant measures: see e.g. [7, Chapitre 7, sec. 2.6, Théorème 3], [24, p. 138, Theorem 1] or [31, Theorem 7.4.1 and Corollary 7.4.2].

(c) is now an easy consequence, as we can write (uniquely) $\mu = \sum_{k=0}^{r-1} \mu_k$ with $\mu_k$ supported on $\Omega_k$, and each $\mu_k$ is $G$-relatively invariant with exponent $\kappa$ [since each set $\Omega_k$ is a separate $G$-orbit]. But in at most one case can $\kappa$ take the correct value $kd/2n$; so all but one of the measures $\mu_k$ must be zero. □

**Remarks.** — 1. Assertions (a) and (b) are false when $r = 1$: the determinant $\Delta(x)$ is invariant under the action of $G \cap SL(V)$ [cf. (4)], so $G \cap SL(V)$ cannot act transitively on $\Omega_r$; and all the measures $\mathcal{R}_\alpha$ with $\Re \alpha > (r-1)/2$ are $G$-relatively invariant [hence $G \cap SL(V)$-invariant] and supported on $\Omega_r$.

2. A slight weakening of Lemma 3.5(b) — in which “$G \cap SL(V)$-invariant” is replaced by “$G$-relatively invariant with some exponent $\kappa$” — is asserted in [21, p. 391, Remarque 3], but the proof given there is insufficient (if it were valid, it

(3) More generally, we see that $P(\sum \lambda_i c_i)$ acts as multiplication by $\lambda_i \lambda_j$ on $V_{ij}$.
would apply also to \( k = r \). However, Michel Lassalle has kindly communicated to me a simple alternative proof of this result, based on \[21, \text{Théorème 3 and Proposition 11(b)}\].

3. Further information on the Riesz measures \( R_{k,d/2} \) for \( 0 \leq k \leq r - 1 \) can be found in \[21, 6\].

**Proof of Theorem 1.2.** — We already know from Proposition 3.3(b) that \( R_\alpha \) is a locally finite complex measure for \( \Re \alpha > (r - 1)\frac{d}{2} \). On the other hand, by applying Proposition 2.3 to \( F(x,\alpha) = \Delta(x)^{\alpha - \frac{d}{2}}/\Gamma_\Omega(\alpha) \) and using Lemma 3.4, we deduce that \( R_\alpha \) is not a locally finite complex measure whenever \( \Re \alpha \leq (r - 1)\frac{d}{2} \) and \( \Gamma_\Omega(\alpha) \neq \infty \). So it remains only to study the values of \( \alpha \) for which \( \Re \alpha \leq (r - 1)\frac{d}{2} \) and \( \Gamma_\Omega(\alpha) = \infty \). For \( \alpha \in \{0, \frac{d}{2}, \ldots, (r - 1)\frac{d}{2}\} \), we know from Proposition 3.3(a) that \( R_\alpha \) is a positive measure. For \( \alpha \in ((0, \frac{d}{2}, \ldots, (r - 1)\frac{d}{2}) \setminus \{0, \frac{d}{2}, \ldots, (r - 1)\frac{d}{2}\}, \) we know from Proposition 3.3(a) and (8c) that \( R_\alpha \) is a distribution supported on \( \partial \Omega \); and by (9) and Lemma 3.5(b) we conclude that it cannot be a locally finite complex measure (here we use the fact that \( R_\alpha \neq R_\beta \) when \( \alpha \neq \beta \)).

**Remark.** — For \( \Re \alpha < 0 \), an alternate proof that \( R_\alpha \) is not a complex measure can be based on the following fact, which is a special case of the \( N = 0 \) case of \[19, \text{Theorem 7.4.3}\] (compare \[19, \text{Theorem 7.3.1}\]) but can also easily be proven by direct computation:

**Lemma 3.6.** — Let \( \Omega \) be a proper open convex cone in a real vector space \( V \), and let \( \Omega^* \subset V^* \) be the open dual cone. Let \( T \in \mathcal{S}'(V) \cap D^0(V) \) be a tempered distribution of order 0 (i.e. a polynomially bounded complex measure) that is supported in \( \Omega \). Then the Laplace transform \( \mathcal{L}T \) is analytic in the complex tube \( \Omega^* + iV^* \) and is bounded in every set \( K + \Omega^* + iV^* \) where \( K \) is a compact subset of \( \Omega^* \).

It then follows from (10) that \( R_\alpha \) cannot be a locally finite complex measure when \( \Re \alpha < 0 \), because \( \Delta(y)^{-\alpha} \) is unbounded at infinity. This argument handles (without the need for Lemma 3.5) the cases \( d = 1 \) (real symmetric matrices) and \( d = 2 \) (complex hermitian matrices) in Theorem 1.2.

**Appendix A**

**Remarks on an elementary proof of Theorem 1.1**

Casalis and Letac \[9, \text{Proposition 5.1}\] have given an elementary proof of Theorem 1.1 that deserves to be more widely known than it apparently is.\(^{(4)}\)

\(^{(4)}\) Science Citation Index shows only ten publications citing \[9\], and six of these have an author in common with \[9\].
They employ a method due to Shanbhag [27, p. 279, Remark 3] — who proved Theorem 1.1 for the cases of real symmetric and complex hermitian matrices — which they abstract as a general “Shanbhag principle” [9, Proposition 3.1]. Here I would like to abstract their method even further, with the aim of revealing its utter simplicity and beauty.

Let $V$ be a finite-dimensional real vector space, and let $V^*$ be its dual space. We then make the following trivial observations:

(a) If $\mu$ is a positive (i.e. nonnegative) measure on $V$, then its Laplace transform

\[ L(\mu)(y) = \int e^{-\langle y,x \rangle} d\mu(x) \]

is nonnegative on any subset of $V^*$ where it is well-defined (i.e. where the integral is convergent).

(b) If $\mu$ is a positive measure on $V$, then so is $f\mu$ for every continuous (or even bounded measurable) function $f$ on $V$ that is nonnegative on $\text{supp} \mu$.

(c) If $\mu$ is a (positive or signed) measure on $V$ whose Laplace transform is well-defined (and finite) on a nonempty open set $\Theta \subseteq V^*$, then the same is true for $P\mu$, where $P$ is any polynomial on $V$; furthermore, $L(P\mu) = P(-\partial) L(\mu)$.(5)

Putting together these observations, we conclude:

**Proposition A.1 (Shanbhag–Casalis–Letac principle)**

If $\mu$ is a positive measure on $V$ whose Laplace transform is well-defined (and finite) on a nonempty open set $\Theta \subseteq V^*$, and $P$ is a polynomial on $V$ that is nonnegative on $\text{supp} \mu$, then $P(-\partial) L(\mu) \geq 0$ everywhere on $\Theta$.

**Remark.** — Proposition A.1 also has a strong converse, which we shall state and prove at the end of this appendix.

Using Proposition A.1, we can give the following slightly simplified version of the Shanbhag–Casalis–Letac argument:

**Proof of Theorem 1.1.** — (Based on [9, Proposition 5.1].) In view of Proposition 3.3, it suffices to prove the converse statement. So let $\alpha \in \mathbb{R}$ and suppose that $R_\alpha$ is a positive measure. Using Proposition A.1 with $P = \Delta$ together with the Laplace-transform formula (10), we conclude that

\[ \Delta (-\partial / \partial y) \Delta(y)^{-\alpha} \geq 0 \quad \text{for all } y \in \Omega . \]

(5) Indeed, the same holds if the measure $\mu$ is replaced by a distribution $T \in \mathcal{D}'(V)$. See [26, Chapitre VIII] or [19, Section 7.4] for the theory of the Laplace transform on $\mathcal{D}'(V)$.
But the “Cayley” identity [12, Proposition VII.1.4] tells us that

\[ \Delta(\partial/\partial y) \Delta(y)^\lambda = \Delta(y)^{\lambda-1} \prod_{j=0}^{r-1} \left( \lambda + j \frac{d}{2} \right), \]

hence (since \( \Delta \) is homogeneous of degree \( r \))

\[ \Delta(-\partial/\partial y) \Delta(y)^{-\alpha} = \Delta(y)^{-\alpha-1} \prod_{j=0}^{r-1} \left( \alpha - j \frac{d}{2} \right). \]

It follows from (20) and (22) that \( R_\alpha \) is not a positive measure when \((r - 2) \frac{d}{2} < \alpha < (r - 1) \frac{d}{2}\). But using the convolution equation (8b) with \( \beta = d/2 \) together with the fact that \( R_{d/2} \) is a positive measure [Proposition 3.3(a)], we conclude successively that \( R_\alpha \) is not a positive measure when \((k - 1) \frac{d}{2} < \alpha < k \frac{d}{2}\) for any integer \( k \leq r - 1 \). This leaves only negative multiples of \( d/2 \); and the argument given after Lemma 3.6 shows that \( R_\alpha \) is not a positive measure whenever \( \alpha < 0 \). (6)

Remarks. — 1. This method has been used recently by Letac and Massam [22, proof of Proposition 2.3] to determine the set of acceptable powers \( p \) for the noncentral Wishart distribution, generalizing the earlier proof of Shanbhag [27] and Casalis and Letac [9] for the ordinary Wishart distribution (which is essentially Theorem 1.1).

2. A very different proof of Theorem 1.1 for the cases \( d = 1, 2 \), using zonal polynomials, was given by Peddada and Richards [25, Theorems 1 and 3].

But this is not yet the end of the story; the proof can be further simplified. The use of the Laplace transform in the foregoing proof is in reality a red herring, as it is used twice in opposite directions: once in the proof of Proposition A.1, and once again in the proof of (21). (7) We can therefore give a direct proof that makes almost no reference to the Laplace transform:

**Second proof of Theorem 1.1.** — Consider first \((r - 2) \frac{d}{2} < \alpha < (r - 1) \frac{d}{2}\). If \( R_\alpha \) is a positive measure, then so is \( \Delta(x) R_\alpha \), which by (8d) equals \( C_\alpha R_{\alpha+1} \).

(6) Alternate argument: For \( k = 1, 2, 3, \ldots \) we know from Proposition 3.3(a,b) and (9) that \( R_{kd/2} \) is a positive measure that is not supported on a single point. If \( R_{-kd/2} \) were a positive measure (recall that we know it is nonzero), then \( R_{kd/2} \ast R_{-kd/2} \) could not be supported on a single point, contrary to the fact that \( R_{kd/2} \ast R_{-kd/2} = \delta \) [cf. (8a,b)].

(7) The simplest proof of (21) is probably the one given in [12, Proposition VII.1.4], using Laplace transforms. However, direct combinatorial proofs are also possible: see [8] for a detailed discussion in the cases of real symmetric and complex hermitian matrices.
where

\[ C_\alpha = \prod_{j=0}^{r-1} (\alpha - j \frac{d}{2}) < 0. \]

It follows that \( R_{\alpha+1} \) must be a negative (i.e. nonpositive) measure. But this is surely not the case, as the Laplace-transform formula (10) immediately implies that no \( R_\beta \) can be a negative measure. This shows that \( R_\alpha \) is not a positive measure when \( (r-2)\frac{d}{2} < \alpha < (r-1)\frac{d}{2} \). The proof is then completed as before.\(^{(8)}\)

It would be interesting to know whether this approach is powerful enough to handle the multiparameter Riesz distributions [12, Theorem VII.3.2] and/or the Riesz distributions on homogeneous cones that are not symmetric and hence do not arise from a Euclidean Jordan algebra [13, 20].

To conclude, let us give the promised strong converse to Proposition A.1:

**Proposition A.2.** — Let \( T \in \mathcal{D}'(V) \) be a distribution whose Laplace transform is well-defined on a nonempty open set \( \Theta \subseteq V^* \). Let \( S \subseteq V \) be a closed set, and suppose that there exists \( y_0 \in \Theta \) such that \( \{ P(-\partial) \mathcal{L}(T) \}(y_0) \geq 0 \) for all polynomials \( P \) on \( V \) that are nonnegative on \( S \). Then \( T \) is in fact a positive measure that is supported on \( S \).

**Proof.** — By replacing \( T(x) \) by \( e^{-\langle y_0, x \rangle} T(x) \), we can assume without loss of generality that \( y_0 = 0 \). Then the derivatives of \( \mathcal{L}(T) \) at the origin give us the moments of \( T \); and the hypothesis \( \{ P(-\partial) \mathcal{L}(T) \}(y_0) \geq 0 \) implies, by Haviland’s theorem [16, 17] [23, Theorem 3.1.2], that there exists a positive measure \( \mu \) supported on \( S \) that has these moments. Furthermore, the analyticity of \( \mathcal{L}(T) \) in the open set \( \Theta + iV^* \) implies that these moments satisfy a bound of the form \( |c_n| \leq AB^{|n|} n! \), so that \( \int e^{\epsilon|x|} \, d\mu(x) < \infty \) for some \( \epsilon > 0 \). It follows that the Laplace transform \( \mathcal{L}(\mu) \) is well-defined and analytic in a neighborhood of the origin; and since its derivatives at the origin agree with those of \( \mathcal{L}(T) \), we must have \( \mathcal{L}(\mu) = \mathcal{L}(T) \). But by the injectivity of the distributional Laplace transform [26, p. 306, Proposition 6], it follows that \( \mu = T \). \( \square \)

\(^{(8)}\) It would be interesting to know whether this residual use of the Laplace transform can be avoided. For \( d \leq 2 \) it can definitely be avoided, as \( \alpha + 1 > (r-1)\frac{d}{2} \), so that \( R_{\alpha+1} \) is a nonzero positive measure by Proposition 3.3(b); but for \( d > 2 \) I do not know.

\(^{(9)}\) The argument given after Lemma 3.6 explicitly uses the Laplace transform. But the alternate argument given in footnote 6 does not.
In Proposition A.2 it is essential that the Laplace transform of $T$ be well-defined on a nonempty open set $\Theta \ni y_0$, or in other words (when $y_0 = 0$) that $T$ have some exponential decay at infinity [in the sense that $\cosh(\epsilon|x|)T \in \mathcal{S}'$ for some $\epsilon > 0$]. It is not sufficient for $T$ to have finite moments of all orders satisfying $T(P) \geq 0$ for all polynomials $P$ on $V$ that are nonnegative on $S$. Indeed, Stieltjes’ [28] famous example

$$f(x) = \begin{cases} e^{-\log^2 x \sin(2\pi \log x)} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

belongs to $\mathcal{S}(\mathbb{R})$ and has zero moments of all orders [i.e. $T(P) = 0$ for all polynomials $P$] but is not nonnegative.

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